



**BIRZEIT UNIVERSITY**

Faculty Of Graduate Studies  
Masters program of MATHEMATICS

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**Semimodules on Semirings**

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*Supervisor:*

Dr. Mohammad Saleh

**M.Sc. Thesis**

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This thesis was successfully defended and approved  
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## Abstract

The aim of this thesis is to study properties of semimodules on semirings and concentrate on some types of semimodules, e.g., *projective semimodules*, *injective semimodules*. We start with providing some basics concerning semimodules and some definitions and properties about 30 *exact sequences and the homo-functor*. Afterwards, in part three, we study some properties of *projective ,injective  $S$ -semimodules* ( *$N$ -projective semimodules,  $e$ -projective semimodules,  $N$ - $e$ -projective semimodules,  $N$ -injective semimodules,  $e$ -injective semimodules,  $N$ - $e$ -injective semimodules*), but we concentrate mostly on projective semimodules.

**Keywords:** *projective semimodule,  $N$ -projective semimodules,  $e$ -projective semimodules, injective semimodule,  $N$ -injective semimodules,  $e$ -injective semimodules,  $N - e$ -injective semimodules, exact sequence.*

# المخلص

الهدف الرئيسي من هذه الرسالة دراسة ما يسمى بشبه الوحدات على شبه الحلقات، والتركيز بشكل خاص على نوعين من شبه الوحدات وهما: وحدات شبه اسقاطية، وحدات شبه الحقن ودراسة بعض المفاهيم المتعلقة بها.

كلمات البحث: وحدات شبه اسقاطية، وحدات شبه الحقن، التسلسل الدقيق.

# Notations

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S-subsemimodule	$\leq$
Isomorphic	$\simeq$
The Bourne relation	$\equiv_N$
The direct summand	$\oplus$
The tensor product	$\otimes$
The congruence class of $m$	$[m]_\rho$
The diagonal relation	$\Delta_M$
The additive identity	$0_S$
The multiplicative identity	$1_S$
$\{0, \infty\}$	$\mathbb{B}$
The cokernel map	$coker(f)$
The cokernel set	$Coker(f)$
The set of $S$ – <i>homomorphisms</i> from $M$ to $N$	$Hom_S(M, N)$
The set of additively idempotent elements	$I^+(S)$
The set of idempotent elements	$I(S)$
The identity map	$id_M$
The set of cancellative elements	$K^+(S)$
The kernel map	$ker(f)$
The kernel set	$Ker(f)$
Subtractive closure	$\bar{L}$
The direct limit	$\lim_{\rightarrow}$
The direct colimit	$\lim_{\leftarrow}$
The set of non-negative rational numbers	$\mathbb{Q}^+$
The set of non-negative real numbers	$\mathbb{R}^+$
The direct sum of $S$	$S^{(\Lambda)}$
The direct product of $S$	$S^\Lambda$
The set of elements which have additive inverse	$V(S)$
Non-negative integers	$\mathbb{Z}^+$
Zeroid	$Z(S)$
Generated semimodule	$Span\{.\}$
The quotient semimodule over congruence relation	$M/\rho$
The quotient semimodule over $N$	$M/N$
The endomorphism semiring	$\mathbf{E}_M$
The category of $S$ -semimodules	$\mathbf{SM}$

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# Chapter 1

## Introduction

In the past few years, many researchers introduced several generalizations of semimodules over semirings, and showed to have many applications in our life (such as, in computer science) as we see these applications in [14], [12], [21], and we find several of these applications in [13] (the main reference of this subject).

During 1981 – 1990 *M. Takahashi* characterized the definitions of semimodules in a series of several papers in this period, he characterized two main ideas: the first one is *tensor products* [25] and the second one is *exact sequences* [23], and his characterizations were used by most of the researchers in this subject in the last century.

In 21st century, many researchers began to use a more definitions of semimodules over semirings specially about the tensor product of semimodules as we see in [20], [5] developed these definitions with the category of semimodules over a commutative semiring.

In 2003, several definitions of exact sequences was introduced as we see in [22], the most of these recent definitions was seen in [1] based on an intensive study of the nature of the category of semimodules over a semiring.

Several papers by Abuhlail, *I'llin*, Katsov prepared special topics of semirings using special in projective, injective and flat semimodules ([1], [2],[3], [15], [17], [18], [19]).

In the recent years, all of congruence-simple left (right) semimodules that are injective have been completely characterized in [2], and ideal-semisimple semirings all

of whose left cyclic semimodules are projective have been investigated in [16]. And through their papers, introduced in addition to the categorical notions of projective, injective and flat semimodules over a semiring, e.g.,  $k$ -projective semimodules,  $i$ -injective semimodules, and  $e$ -projective semimodules, normally projective semimodules,  $e$ -injective semimodules, normally injective semimodules.

In this thesis, we try to clarify what has been done so far and review it in our own way. Hopefully, we can add something new with this promising future..

# Chapter 2

## Preliminaries

In this section, we start with the basic definitions and results required in this thesis. Any notations of definition and results not found in this section can be found in [13].

### 2.1 Basic Definitions and Examples

**Definition 2.1.1.** [13] A **semiring** is  $(S, +, 0, \cdot, 1)$  consisting of a nonempty set  $S$  along with two binary operations ” + ” (addition) and ”  $\cdot$  ” (multiplication) such that:

- (1) the set  $(S, +, 0)$  is commutative monoid with neutral element 0.
- (2) the set  $(S, \cdot, 1)$  is a monoid with neutral element 1.
- (3)  $0 \neq 1$ .
- (4)  $s \cdot 0 = 0 = 0 \cdot s$  for any  $s$  belongs in  $S$ .
- (5) For any  $s_1, s_2, s_3 \in S$  we have that

$$s_1(s_2 + s_3) = s_1s_2 + s_1s_3 \text{ and } (s_1 + s_2)s_3 = s_1s_3 + s_2s_3$$

**Definition 2.1.2.** [13] Let  $(S, +, 0, \cdot, 1)$  be a *semiring*.

- If  $0, 1 \subseteq S' \subseteq S$  and  $S'$  is closed under the two binary operations ”+” and ” $\cdot$ ”, then  $S'$  is a *subsemiring* of  $S$ .
- If the monoid  $(S, \cdot, 1)$  is commutative, then  $S$  is a *commutative semiring*.
- If  $(S \setminus \{0\}, \cdot, 1)$  is a group, then  $S$  is a *division semiring*.
- A commutative division *semiring* is called a *semifield*.
- Let  $s \in S$ , then  $s$  is *additive idempotent element* of  $S$  if and only if  $s + s = s$ .

The set of additive idempotent elements is defined as

$$I^+(S) := \{s \in S \mid s + s = s\}. \quad (1)$$

If  $I^+(S) = S$ , then  $S$  is additively idempotent.

- The set of *multiplicatively idempotent elements* of  $S$  is defined as

$$I^\times(S) := \{s \in S \mid s \cdot s = s\}. \quad (2)$$

If  $I^\times(S) = S$ , then  $S$  is *multiplicatively idempotent*.

- The set of *idempotent elements* of  $S$  is defined as

$$I(S) := I^+(S) \cap I^\times(S). \quad (3)$$

If  $I(S) = S$ , then  $S$  is called *idempotent semiring*.

- Let  $s \in S$ , then  $s$  is called additive inverse, if  $\exists s'$  such that  $s + s' = 0$ . The set of additive inverse of  $S$  is denoted by  $V(S)$ , defined by:

$$V(S) := \{s_1 \in S \mid s_1 + s_2 = 0 \text{ for some } s_2 \in S\}. \quad (4)$$

If  $V(S) = \{0\}$ , then  $S$  is called *zerosumfree*. Notice that  $V(S) = S$  if and only if  $S$  is a ring.

- Let  $s \in S$ , then  $s$  is called *cancellative element*, if for any  $a, b \in S$  such that  $s + a = s + b$ , then  $a = b$ . The set of cancellative elements of  $S$  is denoted by  $K^+(S)$ , defined by:

$$K^+(S) = \{s \in S \mid s + a = s + b \implies a = b \text{ for any } a, b \in S\}$$

if  $K^+(S) = S$ , then  $S$  is called *cancellative semiring*.

- Let  $s \in S$ , then  $s$  is called a zero divisor of  $S$  if  $st = 0$  or  $ts = 0$  for some  $t \in S \setminus \{0\}$ . If  $S$  has no non-zero zero-divisors, then  $S$  is called *entire*.
- If  $a \in S$  is such that  $s + a = a$  for all  $s \in S$ , then  $a$  is called an *infinite element* of  $S$ . If  $S$  has an infinite element, then it is *unique*.
- If  $a \in S$  is an *infinite element* such that  $sa = a = as$  for all  $s \in S \setminus \{0\}$ , then  $a$  is a *strongly infinite element*.
- The *zeroid* of  $S$  is defined as

$$Z(S) = \{z \in S \mid z + s = z \text{ for some } s \in S\}. \quad (5)$$

A semiring  $S$  is called a *zeroic semiring* if  $Z(S) = S$ , otherwise  $S$  is *non-zeroic*. On the other hand,  $S$  is a *plain semiring* if  $Z(S) = \{0\}$ , otherwise  $S$  is *nonplain*.

- A left ideal  $I$  of a *Semiring*  $S$  is a nonempty subset of  $S$  satisfying the following conditions:
  - (1) If  $a, b \in I$  then  $a + b \in I$ .
  - (2) If  $a \in I$  and  $s \in S$  then  $sa \in I$ .
  - (3)  $1 \notin I$ .

A right ideal is defined by the same way as the left ideal is defined. Now a nonempty set  $I$  of  $S$  is called an ideal of  $S$  if and only if it is both left and right ideal. The set of all left ideals of  $S$  is denoted by  $\text{lideal}(S)$ , the set of all right ideals of  $S$  is denoted by  $\text{rideal}(S)$ , and the set of all ideals of  $S$  is denoted by  $\text{ideal}(S)$ .

- Let  $S$  be a semiring, and  $C$  is a nonempty subset of  $S$ , then the set  $SC$  is consisting of all finite sums  $\sum_{i \in I} s_i c_i$ , where  $s_i$  in  $S$  and  $c_i$  in  $C$ .  
Note that the set  $SC$  is the smallest left ideal of  $S$  containing  $C$ .
- Let  $S$  be a semiring, and  $M$  is a left ideal of  $S$  [resp. Right ideal, ideal], then  $M$  is called **finitely generated** if and only if there exists a nonempty subset  $B$  of  $S$  satisfies that  $M = SB$  [resp.  $M = BS$ ,  $M = (B)$ ].
- Let  $S$  be a semiring, and  $N$  is a left ideal of  $S$ , then  $N$  is called **principal** if and only if there exists an element  $b$  of  $S$  such that  $N = Sb$  [resp.  $N = bS$ ,  $N = (b)$ ].

### Examples [13]

- Every ring is a cancellative semiring.
- Every distributive bounded lattice  $\ell = (L, \vee, 1, \wedge, 0)$  is a commutative idempotent semiring and  $1$  is an infinite element of  $\ell$ .
- Let  $R$  be any ring. The set  $\wp = (\text{Ideal}(R), +, 0 \cdot, R)$  of ideals of  $R$  is a zero – sumfree semiring and  $R$  is a strongly infinite element of  $\wp$ .
- The set  $(\mathbb{Z}^+, +, 0, \cdot, 1)$  of non-negative integers is a commutative cancellative zerosumfree entire semiring which is not a ring.

- The set  $(\mathbb{R}^+, +, 0, \cdot, 1)$  of non-negative real numbers is a semifield. The subset  $(\mathbb{Q}^+, +, 0, \cdot, 1)$  of non-negative rational numbers is a subsemifield of  $\mathbb{R}^+$ , and  $\mathbb{Z}^+$  is subsemiring of  $\mathbb{Q}^+$ .
- $M_n(S)$ , the set of all  $n \times n$  matrices over a (zerosumfree) semiring  $S$ , is a (zerosumfree)semiring.
- The Boolean algebra  $\mathbb{B} := \{0, 1\}$  with  $1 + 1 = 1$ ,  $1 \cdot 1 = 1$  is an *idempotent semifield* which is not a field and 1 is a *strongly infinite element* of  $B$ .
- The *max-plus algebra*  $\mathbb{R}_{max,+} := (\mathbb{R} \cup \{\infty\}, max, -\infty, +, 0)$  is an additively idempotent semiring.
- The *min-plus algebra*  $\mathbb{R}_{min,+} := (\mathbb{R} \cup \{\infty\}, min, \infty, +, 0)$  is a additively idempotent semiring.
- The *max-min algebra*  $\mathbb{R}_{max,min} := (\mathbb{R} \cup \{-\infty, \infty\}, max, -\infty, min, \infty)$  is an idempotent semiring and  $\infty$  is the infinite element of  $\mathbb{R}_{max,min}$ .

**Definition 2.1.3.** [13] Let  $S$  be a semiring, and  $K$  is a nonempty subset of  $S$ , then the commutative monoid set  $(K, +, 0_K)$  with a map

$$S \times K \longrightarrow K, (s, k) \rightarrow sk, \quad (\text{called scalar multiplication})$$

is called **left  $S$ -semimodule** if and only if satisfies the following conditions for any  $k, k_1, k_2$  in  $K$  and  $s, s_1, s_2$  in  $S$ :

- (1)  $(s_1 s_2)k = s_1(s_2 k)$ .
- (2)  $s(k_1 + k_2) = sk_1 + sk_2$ .
- (3)  $(s_1 + s_2)k = s_1 k + s_2 k$ .
- (4)  $1_S k = k$ .
- (5)  $s 0_K = 0_K = 0_S k$  ( $0_K \neq 0_S$ ).

If  $M$  is a left  $S$  - *semimodule*, and  $(L, +, 0_M) \leq (M, +, 0)$ , is a submonoid such that  $sl \in L$  for all  $s \in S$  and  $l \in L$ , then  $L$  is an

$S$ -subsemimodule of  $M$  and write  $L \leq M$ .

**Definition 2.1.4.** *Let  $A, B$  are left  $S$ -semimodules of semiring  $S$ , then the function  $\beta : A \longrightarrow B$  is  $S$ -homomorphism if and only if satisfies the following:*

- $\beta(a_1 + a_2) = \beta(a_1) + \beta(a_2)$  for all  $a_1, a_2 \in A$ .
- $\beta(sa) = s(\beta(a))$  for all  $a \in A$  and  $s \in S$ .

The set  $Hom_S(M, N)$  of all  $S$ -homomorphisms from  $M$  to  $N$  is a commutative monoid under the usual addition of maps. The category of left  $S$  – semimodules and  $S$ -homomorphisms is denoted by  ${}_S\mathbf{SM}$ . The category  $\mathbf{SM}_S$  of right  $S$  – semimodules is defined analogously.

**Definition 2.1.5.** *Let  $S$  and  $T$  are semirings, and  $K$  is a left  $S$ -semimodule and a right  $T$ -semimodule satisfies that  $(sk)t = s(kt)$  for any  $s$  in  $S$ ,  $t$  in  $T$  and  $k$  in  $K$ , then  $K$  is an  $(S, T)$ -bisemimodule. The category of  $(S, T)$ -bisemimodules, with arrows being the left  $S$ -homomorphisms right  $T$ -homomorphism, is denoted by  ${}_S\mathbf{SM}_T$ .*

**Definition 2.1.6.** *Let  $S$  be a semiring. A left (resp., right) ideal of  $S$  can be defined as an  $S$ -subsemimodule of  ${}_S\mathbf{S}$  (resp., of  $S_S$ ). A (two-sided) ideal of  $S$  can be defined as an  $(S, S)$ -subbisemimodule of  ${}_S\mathbf{S}_S$ .*

**Definition 2.1.7.** *Let  $S$  be a semiring and  $M$  a left  $S$ -semimodule. The subsets  $I^+(M)$  (resp.,  $V(M), K^+(M), Z(M)$ ) of  $M$  are defined in a way analogous to that defined for the semiring  $S$ , and  $M$  is called an **additively idempotent semimodule** (resp., **zerosumfree semimodule, cancellative semimodule, zeroic semimodule, plain semimodule**) if  $I^+(M) = M$  (resp.,  $V(M) = 0, K^+(M) = M, Z(M) =$*



$M, Z(M) = \{0_M\}$ ).

**Example 2.1.1.** Let  $S$  be a semiring with additive identity  $0$ , and multiplicative identity  $1$ , then  $S$  and  $S^\Lambda$  (where  $\Lambda$  is an index set, and  $S^\Lambda$  is the direct sum of  $S$  over  $\Lambda$ ) are  $(S, S)$ -bisemimodules with left and right actions induced by "  $\cdot$  ".

**Example 2.1.2.** Consider the semiring  $M_2(\mathbb{R}^+)$ . Then

$$E_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{R}^+ \right\} \text{ is left } M_2(\mathbb{R}^+) \text{ - semimodule}$$

and

$$E_2 = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}^+ \right\} \text{ is right } M_2(\mathbb{R}^+) \text{ - semimodule}$$

**Definition 2.1.8.** [13] Let  $S$  be a semiring and  $K$  is a left  $S$ -semimodule and  $Y \leq K$ , then:

- The **subtractive closure** of  $Y$  is defined by

$$\bar{Y} := \{k \in K \mid k + y = y' \text{ for some } y, y' \in Y\}.$$

- $Y$  is called **subtractive** if and only if  $Y = \bar{Y}$ .
- The left  $S$ -semimodule  $K$  is called a **subtractive semimodule** if and only if any  $S$ -subsemimodule  $Y \leq K$  is subtractive.

**Definition 2.1.9.** [13] Let  $S$  be a semiring, then  $S$  is a **left subtractive semiring (right subtractive semiring)** if every left (right) ideal of  $S$  is subtractive.

$S$  is a subtractive semiring if  $S$  is both left and right subtractive.

**Definition 2.1.10.** [13] *Let  $S$  be a semiring, and  $N$  is a left  $S$ -semimodule, then  $\rho$  is called a **congruence relation** if it satisfies the following: for all  $s \in S$  and  $n, n', a, a' \in N$  :*

$$\begin{aligned} n\rho n' \text{ and } a\rho a' &\Rightarrow (n + a')\rho(a + n'), \\ n\rho n' &\Rightarrow (sn)\rho(sn'). \end{aligned}$$

**Example 2.1.3.** *Let  $S$  be a semiring, and  $M$  is a left  $S$ -semimodule and  $N \leq M$ . The **Bourne relation**  $\equiv_N$  on  $M$  is defined as:*

$$m \equiv_N m' \Leftrightarrow m + n = m' + n' \text{ for some } n, n' \in N$$

It is clear that  $\equiv_N$  is a congruence relation. Moreover,  $M/N = M/\equiv_N = \{[m]_N \mid m \in M\} (= M/\overline{N})$  is a left  $S$ -semimodule, the canonical surjective map  $\pi_N : M \rightarrow M/N$  is  $S$ -homomorphism, and  $\text{Ker}(\pi_N) = \overline{N}$ , In particular,  $\text{Ker}(\pi_N) = 0$  if and only if  $N \leq M$  is subtractive (this explains why subtractive ideals are called  $k$ -ideals in many references).

**Definition 2.1.11.** *Let  $S$  be a semiring and  $K$  is a left  $S$ -semimodule, then  $K$  is called:*

- **Ideal-simple** if and only if the only  $S$ -subsemimodules of  $K$  are  $0$  and  $K$ .
- **Congruence-simple** if and only if the only congruence relations on  $K$  are  $\{K \times K, \text{ and } \Delta_K := \{(k, k) \mid k \in k\}\}$ .

**Remark 2.1.1.** *Let  $S$  be a semiring and  $K$  is a congruence-simple left  $S$ -semimodule, then the only subtractive  $S$ -subsemimodules of  $K$  are  $0$  and  $K$ .*

*Proof.* Suppose that  $N \neq 0$  is a subtractive  $S$ -subsemimodule of  $K$ . Then  $\equiv_N$  is a congruence relation on  $K$  with  $n \equiv_N 0$  for some  $n \in N \setminus 0$ . Thus  $\equiv_N \neq \Delta_K$ , which implies  $\equiv_N = K^2$  as  $K$  is congruence-simple. If  $k \in K$ , then  $kK^2 0$ , that is  $k \equiv_N 0$ . Therefore, there exist  $n, n' \in N$  such that  $k + n = n'$ . Since  $N$  subtractive,  $k \in N$ . Hence  $K = N$ . ■

**Example 2.1.4.** [19] *Let  $(M, +, 0)$  be a finite lattice that is not distributive. The endomorphism semiring  $E_M$  of  $M$  is a congruence-simple semiring which is not ideal-simple.*

**Example 2.1.5.** [19] *Every zerosumfree division semiring that is not isomorphic to  $\mathbb{B}$  (e.g.,  $\mathbb{R}^+$ ) is left ideal-simple but not left congruence-simple. Notice that  $D$  is ideal-simple as the only left ideals of  $D$  are  $\{0\}$  and  $D$ . On the other hand, if  $D$  is not isomorphic to  $\mathbb{B}$ . Then*

$$\rho = \{(a, b) \mid a, b \in D \setminus \{0\}\} \cup \{(0, 0)\}$$

*is a non-trivial non-universal congruence relation on  ${}_D D$ , whence  $D$  is not left congruence semisimple.*

**Lemma 2.1.1.** *A left  $S$ -semimodule  $M$  is congruence-simple if and only if every nonzero  $S$ -homomorphism from  $M$  is injective.*

*Proof.* ( $\Rightarrow$ ) Let  $f : M \rightarrow N$  be a non-zero  $S$ -homomorphism and pick some  $m \in M \setminus \{0\}$  such that  $f(m) \neq 0$ . Since  $\equiv_f$  is a congruence relation on  $M$  with  $m \not\equiv_f 0$ , we know  $\equiv_f \neq M^2$ . It follows that  $\equiv_f = \Delta_M$  as  $M$  is congruence-simple. Hence  $f$  is injective.

( $\Leftarrow$ ) Assume that  $M$  is congruence-simple. Let  $\rho$  be a congruence relation on  $M$ . The canonical map  $f : M \rightarrow M/\rho$  is  $S$ -homomorphism. If  $f = 0$ , then  $[m]_\rho = [0]_\rho$  for every  $m \in M$ , that is  $m\rho 0$  for every  $m \in M$  and  $m\rho m'$  for every  $m, m' \in M$ . If  $f \neq 0$ , then  $f$  is injective, that is  $[m]_\rho \neq [m']_\rho$  whenever  $m \neq m'$ . Thus  $m \not\rho m'$  whenever  $m \neq m'$  and  $\rho = \Delta_M$ . ■

**Lemma 2.1.2.** *A left  $S$ -semimodule  $M$  is ideal-simple if and only if every non-zero  $S$ -homomorphism to  $M$  is surjective.*

*Proof.* ( $\implies$ ) Let  $f : L \longrightarrow M$  be a non-zero  $S$ -homomorphism. Then there exists  $l \in L \setminus 0$  such that  $f(l) \neq 0$ . Thus,  $f(L)$  is a non-zero subsemimodule of  $M$  and so  $f(L) = M$  as  $M$  ideal-simple.

( $\impliedby$ ) Let  $K$  be a subsemimodule of  $M$ . Then the injection map  $i : K \longrightarrow M$  is an  $S$ -homomorphism. If  $i = 0$ , then  $K = i(K) = 0$ . If  $i \neq 0$ , then  $i$  is surjective, that is  $K = i(K) = M$ . ■

**Definition 2.1.12.** [13] *Suppose that  $S$  is a semiring and  $N$  is a left  $S$ -semimodule, then  $N$  is called the **direct sum** of a family  $\{K_\lambda\}_{\lambda \in \Lambda}$  of  $S$ -subsemimodules (where  $K_\lambda \leq N$ ), and  $N$  can be written as  $N = \bigoplus_{\lambda \in \Lambda} K_\lambda$ , if for any  $n \in N$  can be written uniquely as a finite sum  $n = k_{\lambda_1} + \dots + k_{\lambda_k}$  where  $k_{\lambda_i} \in K_{\lambda_i}$  for any  $i = 1, \dots, k$ . Equivalently,  $N = \bigoplus_{\lambda \in \Lambda} K_\lambda$  if  $N = \sum_{\lambda \in \Lambda} K_\lambda$  and for each finite subset  $A \subseteq \Lambda$  with  $k_a, k'_a \in K_a$ , we have:*

$$\sum_{a \in A} k_a = \sum_{a \in A} k'_a \implies k_a = k'_a \text{ for all } a \in A.$$

**Definition 2.1.13.** *Let  $S$  be a semiring, then an  $S$ -semimodule  $M$  a **retract** of an  $S$ -semimodule  $N$  if there exist a (surjective)  $S$ -homomorphism  $\phi : N \longrightarrow M$  and an (injective)  $S$ -homomorphism  $\alpha : M \longrightarrow N$  such that  $\phi \circ \alpha = id_M$ .*

**Definition 2.1.14.** *Let  $S$  be a semiring, then an  $S$ -semimodule  $M$  is called a **direct summand** of an  $S$ -semimodule  $N$  if and only if  $N = M \oplus M'$  for some  $S$ -subsemimodule  $M'$  of  $N$ .*

**Remark 2.1.2** *An  $S$ -semimodule  $M$  is a direct summand of an  $S$ -semimodule  $N$  if and only if there exists  $\phi \in \text{Comp}(\text{End}(N_S))$  s.t.  $\phi(N) = M$  where for any semiring  $S$  we set*

$$\text{Comp}(S) = \{s \in S \mid \exists s' \in S \text{ with } s + s' = 1_S \text{ and } ss' = 0_S = s's\}.$$

*Proof.* ( $\implies$ ) Suppose  $M$  is a direct summand of  $N$  and if we have  $i : M \longrightarrow N$  be the inclusion map, then  $M = \pi_M(N)$  where  $\pi_M$  is an endomorphism of  $N$ . Now, if  $N = M \oplus M'$  then  $\pi_M + \pi_{M'} = 1_S$  and  $\pi_M \circ \pi_{M'} = \pi_{M'} \circ \pi_M = 0_S$  so  $\pi_M \in \text{comp}(S)$ .

( $\impliedby$ ) let  $M = \phi(N)$  for some  $\phi \in \text{comp}(S)$ . If  $M' = \phi^\perp(N)$  then it is obvious to verify that  $N = M \oplus M'$ . ■

Now, we have that every direct summand of a  $S$ -semimodule  $N$  is a retract of  $N$ , but in general the converse is not true.

**Remarks 2.1.3.** *Let  $S$  be a semiring,  $C$  is a left  $S$ -semimodule and  $X, Y \leq C$  be  $S$ -subsemimodules of  $C$ , then we have the following:*

(1) *If  $X + Y$  is direct, then  $X \cap Y = 0$ , but the converse is not true.*

(2) *If  $C = X \oplus Y$ , then  $C/X \simeq Y$ .*

The following example proves that the converse of part one in the previous remark is not true.

**Example 2.1.6.** *Let  $S = M_2(\mathbb{R}^+)$ . Notice that*

$$E_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{R}^+ \right\}$$

and

$$N_1 = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mid a \leq c, b \leq d, \text{ where } a, b, c, d \in \mathbb{R}^+ \right\}$$

are left ideals of  $S$  with  $E_1 \cap N_1 = \{0\}$ . However, the sum  $E_1 + N_1$  is not direct since

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

**Definition 2.1.15.** [13] Suppose that  $S$  is a semiring, then a left  $S$ -semimodule  $N$  is called :

- **ideal-semisimple** if it is a direct sum of ideal-simple  $S$ -subsemimodules ( $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$ ).
- **Congruence-semisimple** if it is a direct sum of congruence-simple Subsemimodules ( $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ ).

Suppose that  $S$  is a semiring and  $N$  is a left  $S$ -semimodule, then for a nonempty subset  $B$  of  $N$  there exists an  $S$ -homomorphism  $\phi : S^{(B)} \rightarrow N$  defined by  $\phi : g \mapsto \sum_{m \in B} g(m)m$ . Note that the set  $B$  is a set of generators for  $N$  precisely if  $\phi$  is surjective. Moreover,  $\phi$  induces an  $S$ -congruence relation  $\equiv_\phi$  on  $S^{(B)}$  as defined as  $\phi \circ g = \phi(f)$ .

- The set  $B$  is called **linearly independent** if and only if  $\sum_{m \in B} f(m)m = \sum_{m \in B} g(m)m$  implies that  $f = g$  (the trivial relation).
- The set  $B$  is called **linearly dependent** If it is not linearly independent.
- The nonempty subset  $B$  of  $N$  is called a **basis** of  $N$  over  $S$  if and only if it is linearly-independent set of generators for  $N$ .

**Definition 2.1.16.** [13] *Let  $S$  be a semiring, then a left  $S$ -semimodule  $M$  is called **free**  $S$ -semimodule if and only if it has a basis over  $S$ .*

Let  $S$  be a ring and  $N$  is a left  $S$  – module, then the definition of free  $R$ -module is the same as in case of  $S$ -semimodule. Since not every module over a ring is free, certainly not every semimodule over a semiring is free. As a result of definitions, we note that if  $B$  is a nonempty set, then the left  $S$  – semimodule  $S^{(B)}$  is free, and that for every free left  $S$  – semimodule is  $S$  – isomorphic to  $S^{(B)}$  for some suitable nonempty set  $B$ .

**Proposition 2.1.3.** [13] *Suppose that  $S$  is a semiring and  $N$  is a left  $S$ -semimodule then there exists a free  $S$ -semimodule  $M$  and a surjective  $S$ -homomorphism from  $M$  to  $N$ .*

*Proof.* Suppose  $N$  is a left  $S$  – semimodule, then we have two cases:  
Case (1) if  $N = \{0\}$ , we have done.

Case (2) if  $N \neq \{0\}$ . Let  $N' = N \setminus \{0\}$  and let  $M = S^{(N')}$ . Let  $\phi : M \rightarrow N$  is defined by  $\phi : g \mapsto \sum_{m \in \text{supp}(g)} g(m)$ . This is obvious a surjective  $S$ -homomorphism. ■

**Proposition 2.1.4.** [13] *Let  $S$  be a semiring and  $N$  is a free left  $S$ -semimodule with basis  $B$  and let  $M$  be an arbitrary left  $S$ -semimodule. Now for any function  $f \in M^B$ , then there is a unique  $S$ -homomorphism  $\phi : N \rightarrow M$  satisfy that  $\phi(b) = f(b)$  for any  $b \in B$ .*

*Proof.* Since  $N$  is free, then We can write any element  $n$  of  $N$  uniquely in the form  $\sum_{b \in B} s_b b$ , where  $s_b \in S$  only finitely –  $n$  any of which are nonzero. Let  $\phi : N \rightarrow M$  be a function is defined by  $\sum s_b b \mapsto \sum s_b f(b)$ . It is obvious to verify that  $\phi$  is an  $S$ -homomorphism. Moreover, if  $\alpha : N \rightarrow M$  is an  $S$  – homomorphism satisfying that  $\alpha(b) = f(b)$  for any  $b \in B$ , then  $\alpha(\sum s_b b) = \sum s_b(\alpha(b)) = \sum s_b f(b) = \sum s_b \alpha(b) = \alpha(\sum s_b b)$  and so  $\phi = \alpha$ , so  $\phi$  is unique. ■

**Remark 2.1.4** *If  $S$  is a semiring, and  $N$  is a left  $S$ -semimodule,  $B$  is a nonempty set, and if  $f$  is a function from  $B$  to  $N$ , then there exists a unique  $S$ -homomorphism  $\phi : S^{(B)} \rightarrow N$  such that  $\phi : g_b \mapsto f(b)$ , where  $g_b \in S^B$  for any  $b \in B$ .*

## 2.2 Exact Sequences

During this work,  $(S, +, 0, \cdot, 1)$  is a semiring.

**Definition 2.2.1.** *Let  $N$  and  $M$  are  $S$ -semimodules, then the morphism map  $\phi : M \rightarrow N$  is called*

- **k-normal** if whenever  $\phi(n) = \phi(n')$  for some  $n, n' \in N$ , then we have  $n + k = n' + k'$  where  $k, k' \in \text{Ker}(\phi)$ .
- **i-normal** if  $\text{Im}(\phi) = \overline{\phi(M)} (:= \{n \in N \mid n + m \in M \text{ for some } m \in M\})$ .
- **Normal** if and only if the map  $\phi$  satisfies the both properties (k-normal and i-normal).

**Remarks** (1) *If we see in [23] and [13], we find that the definitions of k-normal (resp., i-normal, normal)  $S$ -homomorphisms are called k-regular (resp., i-regular, regular) morphisms. So we changed them to avoid any confusion about the definitions of regular monomorphisms and regular epimorphisms when applied to categories of semimodules, because they will have different meanings in this case.*

(2) *every surjective  $S$ -homomorphism is i-normal, whence the k-normal surjective  $S$ -homomorphisms are normal and are precisely the so-called **normal epimorphisms**. On the other hand, the injective  $S$ -homomorphisms*



are  $k$ -normal, whence the  $i$ -normal injective  $S$ -homomorphisms are normal and are precisely the so called **normal monomorphisms** see [1].

**Lemma 2.2.1.** *Suppose that  $L$ ,  $M$ , and  $K$  are  $S$ -semimodules, and  $K \xrightarrow{h} L \xrightarrow{f} M$  be a sequence of semimodules.*

(1) *Suppose that  $f$  is injective, then we have the following:*

- (a)  *$h$  is  $k$ -normal if and only if  $f \circ h$  is  $k$ -normal.*
- (b)  *$f \circ h$  is  $i$ -normal(normal), then  $h$  is  $i$ -normal(normal).*
- (c) *Assume that  $f$  is  $i$ -normal. Then  $h$  is  $i$ -normal(normal) if and only if  $f \circ h$  is  $i$ -normal(normal).*

(2) *Suppose that  $h$  be surjective, then we have the following:*

- (a) *If  $f$  is  $i$ -normal if and only if  $f \circ h$  is  $i$  – normal.*
- (b) *If  $f \circ h$  is  $k$  – normal(normal), then  $f$  is  $k$  – normal(normal).*
- (c) *Assume that  $h$  is  $k$  – normal. Then  $f$  is  $k$  – normal(normal) if and only if  $f \circ h$  is  $k$  – normal(normal).*

*Proof.* (1) Let  $f$  be injective: in particular,  $f$  is  $k$ -normal.

(a) Suppose that  $h$  is  $k$  – normal. Assume  $(f \circ h)(k_1) = (f \circ h)(k_2)$  for some  $k_1, k_2 \in K$ . Since  $f$  is injective,  $h(k_1) = h(k_2)$ . By assumption, then there exist  $b_1, b_2$  belongs to  $Ker(h)$  such that  $k_1 + b_1 = k_2 + b_2$ . Since  $Ker(h) \subseteq Ker(f \circ h)$ , so we have  $f \circ h$  is  $k$ -normal. Moreover, suppose that  $f \circ h$  is  $k$  – normal. Assume that  $h(k_1) = h(k_2)$  for some  $k_1, k_2$  belongs to  $K$ . Then  $(f \circ h)(k_1) = (f \circ h)(k_2)$  so there exist  $b_1, b_2$

belongs to  $Ker(f \circ h)$  such that  $k_1 + b_1 = k_2 + b_2$ . Since  $f$  is *injective*,  $Ker(f \circ h) = Ker(h)$  whence  $h$  is *k-normal*.

(b) Suppose that  $f \circ h$  is *i-normal*. Let  $l \in \overline{h(K)}$ , so we have  $l + h(k_1) = h(k_2)$  for some  $k_1, k_2$  belongs to  $L$ . Then we have  $f(l)$  belongs to  $\overline{(f \circ h)(K)} = (f \circ h)(K)$ . Since  $f$  is *injective*,  $l$  belongs to  $h(K)$ . So  $h$  is *i-normal*.

(c) Suppose that  $f$  and  $h$  are *i-normal*. Now Let  $m$  belongs to  $\overline{(f \circ h)(K)}$ , so we have  $m + f(h(k_1)) = f(h(k_2))$  for some  $k_1, k_2$  belongs to  $K$ . Since  $f$  is *i-normal*,  $m$  belongs to  $f(L)$  say  $n = f(l)$  for some  $l \in L$ . Not that  $f$  is *injective*, whence  $l + h(k_1) = h(k_2)$ , (i.e.  $l \in \overline{h(K)} = h(K)$  since  $h$  is *i-normal* by assumption, so  $n = f(l) \in (f \circ h)(K)$ . We conclude  $f \circ h$  is *i-normal*).

(2) Let  $h$  be *surjective*, in particular,  $h$  is *i-normal*.

(a) Suppose that  $f$  is *i-normal*. Let  $m$  belongs to  $\overline{(f \circ h)(K)}$ , so  $m + f(h(k_1)) = f(h(k_2))$  for some  $k_1, k_2$  belongs to  $L$ . Since  $f$  is *i-normal*,  $m = f(l)$  for some  $l$  belongs to  $L$ . Since  $h$  is *surjective*,  $m = f(l) \in (f \circ h)(K)$ , so  $f \circ h$  is *i-normal*. Moreover, suppose  $f \circ h$  is *i-normal*. Now suppose  $m$  belongs to  $\overline{f(L)}$ , so  $m + f(l_1) = f(l_2)$  for some  $l_1, l_2$  belongs to  $L$ . Since  $h$  is *surjective*, there exist  $k_1, k_2 \in K$  such that  $h(k_1) = l_1$  and  $h(k_2) = l_2$ . Then  $m + (f \circ h)(k_1) = (f \circ h)(k_2)$ , i.e. ( $m$  belongs to  $\overline{(f \circ h)(K)} = (f \circ h)(K) \subseteq f(L)$ ). So  $f$  is *i-normal*.

(b) Suppose  $f \circ h$  is *k-normal*. Assume  $f(l_1) = f(l_2)$  for some  $l_1, l_2$  belongs to  $L$ . Since  $h$  is *surjective*, then  $(f \circ h)(k_1) = (f \circ h)(k_2)$  for some  $k_1, k_2$  belongs to  $K$ . By assumption,  $f \circ g$  is *k-normal* and so there exist  $b_1, b_2$  belongs to  $Ker(f \circ h)$  such that  $k_1 + b_1 = k_2 + b_2$  whence  $l_1 + h(b_1) = l_2 + h(b_2)$ . Indeed,  $h(b_1), h(b_2)$  belongs to  $Ker(f)$ . i.e. ( $f$  is *k-normal*).

(c) Suppose  $h$  and  $f$  are  $k$ -normal. Assume  $(f \circ h)(k_1) = (f \circ g)(k_2)$  for some  $k_1, k_2$  belongs to  $K$ . Since  $f$  is  $k$ -normal, then  $h(k_1) + b_1 = h(k_2) + b_2$  where  $b_1, b_2$  belongs to  $\text{Ker}(f)$ . But,  $h$  is surjective, whence  $b_1 = h(l'_1)$  and  $b_2 = h(k'_2)$  where  $k'_1, k'_2$  belongs in  $K$ , i.e.  $h(k_1 + k'_1) = h(k_2 + k'_2)$ . Since  $h$  is  $k$ -normal,  $k_1 + k'_1 + b'_1 = k_2 + k'_2 + b'_2$  where  $b'_1, b'_2$  belongs to  $\text{Ker}(h)$ . Indeed,  $k'_1 + b'_1, k'_2 + b'_2$  belongs in  $\text{Ker}(f \circ h)$ . We conclude  $f \circ h$  is  $k$ -normal. ■

**Lemma 2.2.2.** (1) Let  $\{f_\lambda : L_\lambda \longrightarrow M_\lambda\}_\Lambda$  be a family of left  $S$ -semimodule morphisms and consider the induced  $S$ -homomorphism  $f : \bigoplus_{\lambda \in \Lambda} L_\lambda \implies \bigoplus_{\lambda \in \Lambda} M_\lambda$ . Then  $f$  is normal (resp.  $k$ -normal,  $i$ -normal) if and only if  $f_\lambda$  is normal (resp.  $k$ -normal,  $i$ -normal) for every  $\lambda \in \Lambda$ .

(2) A morphism  $\varphi : L \longrightarrow M$  of left  $S$ -semimodules is normal (resp.  $k$ -normal,  $i$ -normal) if and only if  $\text{id}_F \otimes_S \varphi : F \otimes_S L \longrightarrow \otimes_S M$  is normal (resp.  $k$ -normal,  $i$ -normal) for every non-zero free right  $S$ -semimodule  $F$ .

(3) If  $P_S$  is projective and  $\varphi : L \longrightarrow M$  is a normal (resp.  $k$ -normal,  $i$ -normal) morphism of left  $S$ -semimodules, then  $\text{id}_F \otimes_S \varphi : P \otimes_S L \longrightarrow P \otimes_S M$  is normal (resp.  $k$ -normal,  $i$ -normal).

**Definition 2.2.2.** [1] Let  $X, Y$  and  $Z$  be a left  $S$ -semimodules, then the following sequence of left  $S$ -semimodules

$$X \xrightarrow{\beta} Y \xrightarrow{\phi} Z \quad (6)$$

is exact if  $\phi$  is  $k$ -normal and  $\beta(Y) = \text{Ker}(\phi)$ .

**Definition 2.2.3.** Let  $X, Y$  and  $Z$  be a left  $S$ -semimodules, then the following sequence of left  $S$ -semimodules

$$X \xrightarrow{\beta} Y \xrightarrow{\phi} Z$$

is called:

- **Proper-exact** if  $\beta(X) = \text{Ker}(\phi)$ .
- **Semi-exact** if  $\overline{\beta(X)} = \text{Ker}(\phi)$ .
- **Quasi-exact** if  $\overline{\beta(X)} = \text{Ker}(\phi)$  and  $\phi$  is  $k$ -normal.

**Definition 2.2.4.** A (possibly infinite) sequence of  $S$ -semimodules

$$\dots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \rightarrow \dots \quad (7)$$

- **Chain complex** if  $f_{j+1} \circ f_j = 0$  for every  $j$ .
- **Exact** (resp., proper-exact, semi-exact, quasi-exact) if each partial sequence with three terms  $M_j \xrightarrow{f_j} M_{j+1} \xrightarrow{f_{j+1}} M_{j+2}$  is exact (resp., proper-exact, semi-exact, quasi-exact).

**Remark 2.2.1.** Note that In (6), the inclusion  $\beta(X) \subseteq \text{Ker}(\phi)$  forces  $\beta(X) \subseteq \overline{\beta(X)} \subseteq \text{Ker}(\phi)$ , whence the assumption  $\beta(X) = \text{Ker}(\phi)$  so we have that  $\beta(X) = \overline{\beta(X)}$ , i.e.  $\phi$  is  $i$ -normal. So, the definition puts conditions on  $h$  and  $f$  that are dual to each other (in some sense).

**Lemma 2.2.3.** Suppose that  $X$ ,  $Y$  and  $Z$  are  $S$ -semimodules, then:

- (1)  $0 \rightarrow X \xrightarrow{\beta} Y$  is exact sequence if and only if  $\beta$  is injective.
- (2)  $Y \xrightarrow{\phi} Z \rightarrow 0$  is exact sequence if and only if  $\phi$  is surjective.
- (3)  $0 \rightarrow X \xrightarrow{\beta} Y \xrightarrow{\phi} Z$  is semi-exact sequence and  $\phi$  is normal if and only if  $X \simeq \text{Ker}(\phi)$ .

(4)  $0 \longrightarrow X \xrightarrow{\beta} Y \xrightarrow{\phi} Z$  is exact sequence if and only if  $X \simeq \text{Ker}(\phi)$  and  $\phi$  is  $k$ -normal.

(5)  $X \xrightarrow{\beta} Y \xrightarrow{\phi} Z \longrightarrow 0$  is semi-exact sequence and  $\phi$  is normal if and only if  $Z \simeq Y/\beta(X)$ .

(6)  $X \xrightarrow{\beta} Y \xrightarrow{\phi} Z \longrightarrow 0$  is exact sequence if and only if  $Z \simeq Y/\phi(X)$  and  $\beta$  is  $i$ -normal.

(7)  $0 \longrightarrow X \xrightarrow{\beta} Y \xrightarrow{\phi} Z \longrightarrow 0$  is exact sequence if and only if  $X \simeq \text{Ker}(\beta)$  and  $Z \simeq Y/X$

**Corollary 2.2.1.** *The following are equivalent:*

(1)  $0 \longrightarrow K \xrightarrow{h} L \xrightarrow{f} M \longrightarrow 0$  is an exact sequence of  $S$ -semimodules.

(2)  $K \simeq \text{Ker}(f)$  and  $M \simeq L/K$ .

(3)  $h$  is injective,  $h(K) \simeq \text{Ker}(f)$ ,  $g$  is surjective and  $(k)$ -normal.

In this case We have that  $f, g$  are normal morphism.

**Remark 2.2.2.** *Let  $A, B$  are left  $S$ -semimodules, then  $A$  morphism of semimodules  $\alpha : A \longrightarrow B$  is an isomorphism if and only if*

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

*is exact if and only if  $\alpha$  is a normal bimorphism ( $\alpha$  is both normal monomorphism and normal epimorphism).*

Now the hypothesis about  $\alpha$  that it is normal is necessary. For example, Let  $q : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ , where  $q$  is bimorphism of commutative monoids

( $\mathbb{Z}^+$ -semimodules), but  $q$  is not isomorphism (note that  $q$  is not isomorphism, in fact  $q(\mathbb{Z}^+) = \mathbb{Z}$ ).

**Remark 2.2.3.** (1) An  $S$ -homomorphism is called a monomorphism if and only if it is injective.

(2) Every surjective  $S$ -homomorphism is an epimorphism, but the converse is not always true.

The following example shows that not every epimorphism is surjective.

**Example 2.2.1.** Let  $q : \mathbb{Z}^+ \rightarrow \mathbb{Z}$  is a monoid epimorphism as  $(f \circ q)(1_{\mathbb{Z}^+}) = (g \circ q)(1_{\mathbb{Z}^+})$  implies  $f(1_{\mathbb{Z}}) = g(1_{\mathbb{Z}})$  and  $f = g$  for every monoid morphisms  $f, g : \mathbb{Z} \rightarrow M$ . However, it is clear that  $q$  is not surjective.

## 2.3 Adjoint Pairs of Functors

**Definition 2.3.1.** An  $S$ -homomorphism  $h : M \rightarrow N$  is called a **equalizer** of  $f, g : N \rightarrow L$  if  $f \circ h = g \circ h$  and whenever an  $S$ -homomorphism  $h' : M' \rightarrow N$  satisfies  $f \circ h' = g \circ h'$ , there exists a unique  $S$ -homomorphism  $\varphi : M' \rightarrow M$  such that  $h \circ \varphi = h'$

$$\begin{array}{ccccc}
M & \xrightarrow{h} & N & \xrightarrow[f]{g} & L \\
\uparrow \exists! \varphi & & \nearrow h' & & \\
M' & & & & 
\end{array}$$

**Definition 2.3.2.** An  $S$ -homomorphism  $h : M \rightarrow N$  is called a **co-equalizer** of  $f, g : L \rightarrow M$  if  $h \circ f = h \circ g$  and whenever an  $S$ -homomorphism  $h' : M \rightarrow N'$  satisfies  $h' \circ f = h' \circ g$ , there exists a unique  $S$ -homomorphism  $\varphi : N \rightarrow N'$  such that  $\varphi \circ h = h'$

$$\begin{array}{ccccc}
L & \xrightarrow[f]{g} & M & \xrightarrow{h} & N \\
& & \searrow h' & & \downarrow \exists! \varphi \\
& & & & N'
\end{array}$$

**Definition 2.3.3.** Let  $\phi : X \rightarrow Y$  be an  $S$ -homomorphism.  $\mathbf{Ker}(\phi) := \{x \in X \mid \phi(x) = 0\}$ . The map  $\ker(\phi) : \mathbf{Ker}(\phi) \rightarrow X$  is the equalizer of  $\phi$  and the zero map.  $\mathbf{Coker}(\phi) := Y \setminus \phi(X)$ . The map  $\text{coker}(\phi) : Y \rightarrow \mathbf{Coker}(\phi)$  is the co-equalizer of  $\phi$  and the zero map.

**Proposition 2.3.1.** [8] Let  $\mathbb{A}, \mathbb{B}$  be an arbitrary categories and  $\mathbb{A} \xrightarrow{C} \mathbb{B} \xrightarrow{D} \mathbb{A}$  be functors such that  $(C, D)$  is an adjoint pair, then:

- (1)  $C$  preserves all colimits which turn out to exist in  $\mathbb{A}$ .
- (2)  $D$  preserves all limits which turn out to exist in  $\mathbb{B}$ .

**Corollary 2.3.1.** *Suppose that  $S, R$  are semirings and  ${}_R F_S$  a  $(R, S)$ -bisemimodule.*

(1)  $F \otimes_S - : {}_S \mathbf{SM} \rightarrow {}_R \mathbf{SM}$  preserves all colimits.

(a) *Let the set  $\{A_\lambda\}_\Lambda$  be an arbitrary family of left  $S$ -semimodules, then we have a canonical isomorphism of left  $R$ -semimodules*

$$F \otimes_S \bigoplus_{\lambda \in \Lambda} A_\lambda \simeq \bigoplus_{\lambda \in \Lambda} (F \otimes_S A_\lambda).$$

(b) *Let the set  $(A_j, \{h_{jj'}\})_J$  be an arbitrary directed system of left  $S$ -semimodules, then we have an isomorphism of left  $R$ -semimodules*

$$F \otimes_S \varinjlim A_j \simeq \varinjlim (F \otimes_S A_j)$$

(c)  $F \otimes_S -$  preserves coequalizers.

(d)  $F \otimes_S -$  preserves cokernels.

(2)  $\text{Hom}_T(F, -) : {}_T \mathbf{SM} \rightarrow {}_S \mathbf{SM}$  preserves all limits.

(a) *Let the set  $\{B_\lambda\}_{\lambda \in \Lambda}$  be an arbitrary family of left  $R$ -semimodules, then we have a canonical isomorphism of left  $S$ -semimodules*

$$\text{Hom}_R(F, \prod_{\lambda \in \Lambda} B_\lambda) \simeq \prod_{\lambda \in \Lambda} \text{Hom}_R(F, B_\lambda).$$

(b) *Let the set  $(A_j, \{h_{jj'}\})_J$  be an arbitrary inverse system of left  $R$ -semimodules, then we have an isomorphism of left  $S$ -semimodules*

$$\text{Hom}_R(F, \varprojlim A_j) \simeq \varprojlim \text{Hom}_R(F, A_j).$$

(c)  $\text{Hom}_R(F, -)$  preserves equalizers.

(d)  $\text{Hom}_R(F, -)$  preserves kernels.



(3)  $Hom_R(-, F) : {}_R\mathbf{SM} \rightarrow \mathbf{SM}_S$  preserves all limits.

(a) Let the set  $\{B_\lambda\}_\Lambda$  be an arbitrary family of left  $T$ -semimodules, then we have a canonical isomorphism of right  $S$ -semimodules

$$Hom_R\left(\bigoplus_{\lambda \in \Lambda} B_\lambda, F\right) \simeq \prod_{\lambda \in \Lambda} Hom_R(B_\lambda, F).$$

(b) Let the set  $(A_j, \{h_{jj'}\})_J$  be an arbitrary directed system of left  $T$ -semimodules, then we have an isomorphism of right  $S$ -semimodules

$$Hom_R\left(\varinjlim A_j, F\right) \simeq \varprojlim Hom_R(A_j, F).$$

(c)  $Hom_R(-, F)$  converts coequalizers into equalizers.

(d)  $Hom_R(-, F)$  converts cokernels into kernels.

**Proposition 2.3.2.** Let  $R, S$  are semirings, and  ${}_R A_S$  is  $(R, S)$  – bisemimodule and consider the functor  $Hom_R(A, -) : {}_R\mathbf{SM} \rightarrow {}_S\mathbf{SM}$ . Let

$$0 \longrightarrow K \xrightarrow{h} L \xrightarrow{f} M \quad (8)$$

be a sequence of left  $R$ -semimodules and consider the following sequence of left  $S$ -semimodules

$$0 \longrightarrow Hom_R(A, K) \xrightarrow{(A, h)} Hom_R(A, L) \xrightarrow{(A, f)} Hom_R(A, M). \quad (9)$$

(1) Now if the sequence  $0 \longrightarrow K \xrightarrow{h} L$  is exact and  $h$  is normal, then

$$0 \longrightarrow Hom_R(A, K) \xrightarrow{(A, h)} Hom_R(A, L)$$

is exact and  $(A, h)$  is normal.

(2) If the sequence (8) is semi-exact sequence and  $h$  is normal, then the sequence (9) is semi-exact sequence (proper exact sequence) and  $(A, h)$

is normal.

(3) If the sequence (8) is exact sequence and  $\text{Hom}_R(A, -)$  preserves  $k$ -normal morphisms, then the sequence (9) is exact.

*Proof.* (1) Suppose that the sequence  $0 \longrightarrow K \xrightarrow{h} L$  is exact implies that  $h$  is injective, implies that  $(A, h)$  is injective, which implies that  $0 \longrightarrow \text{Hom}_R(A, K) \xrightarrow{(A, h)} \text{Hom}_R(A, L)$  is exact. Now suppose that  $h$  is normal and take the following short exact sequence of  $S$ -semimodules:

$$0 \longrightarrow K \xrightarrow{h} L \xrightarrow{\pi_L} L/K \longrightarrow 0.$$

Now we have that  $K = \text{Ker}(\pi_K)$ (by Lemma 2.2.3). Now By Corollary 2.3.1 we have that  $\text{Hom}_T(G, -)$  preserves kernels and so  $(A, h) = \text{ker}(A, \pi_K)$  whence normal.

(2) By applying Lemma 2.2.3(3), we have that The semi-exactness of the sequence (10) and the normality of  $h$  are equivalent to  $K \simeq \text{Ker}(f)$ . Now since  $\text{Hom}_T(G, -)$  is preserves kernels, we can deduce that  $\text{Hom}_R(A, K) = \text{Ker}((A, f))$  is equivalent to the semi-exactness of the sequence (9) and the normality of  $(A, g)$ . Now note that

$$(A, h)(\text{Hom}_R(A, K)) = \overline{(A, h)(\text{Hom}_R(A, K))} = \text{Ker}(A, f),$$

that means that the sequence (9) is proper exact sequence(whence semi-exact sequence).

(3) This directly follow from "2" and the hypothesis on  $\text{Hom}_T(-, G)$ . ■

**Proposition 2.3.3.** Let  $R, S$  be as a semirings, and  ${}_R A_S$  be a  $(R, S)$ -bisemimodule and take the functor  $A \otimes_S - : {}_S \mathbf{SM} \longrightarrow {}_R \mathbf{SM}$ . Let

$$X \xrightarrow{h} Y \xrightarrow{\phi} Z \longrightarrow 0 \quad (10)$$

be a sequence of left  $S$ -semimodules and take the sequence of left  $R$ -semimodules

$$A \otimes_S X \xrightarrow{A \otimes h} A \otimes_S Y \xrightarrow{A \otimes \phi} A \otimes_S Z \rightarrow 0 \quad (11)$$

(1) If the sequence  $Y \xrightarrow{\phi} Z \rightarrow 0$  is exact sequence and  $\phi$  is normal, then the sequence  $A \otimes_S Y \xrightarrow{A \otimes \phi} A \otimes_S Z \rightarrow 0$  is exact sequence and  $A \otimes \phi$  is normal.

(2) If the sequence (10) is semi-exact sequence and  $\phi$  is normal, then the sequence (11) is semi-exact sequence and  $A \otimes \phi$  is normal.

(3) If the sequence (10) is exact sequence and  $A \otimes S$ -preserves  $i$ -normal morphisms, then the sequence (11) is exact sequence.

*Proof.* Now The following implications are obvious:  $Y \xrightarrow{\phi} Z \rightarrow 0$  is exact sequence  $\Rightarrow \phi$  is surjective  $\Rightarrow A \otimes \phi$  is surjective  $\Rightarrow A \otimes_S Y \xrightarrow{A \otimes \phi} A \otimes_S Z \rightarrow 0$  is exact.

(1) Now suppose that  $\phi$  is normal and assume the exact sequence of  $S$  – semimodules

$$0 \longrightarrow \text{Ker}(\phi) \xrightarrow{i} Y \xrightarrow{h} Z \longrightarrow 0.$$

Then  $Z \simeq \text{Coker}(i)$ . Now By Corollary 2.3.1(1),  $A \otimes S$ - preserves cokernels and so  $A \otimes \phi = \text{coker}(A \otimes i)$  whence normal.

(2) Apply Lemma 2.2.3: The assumptions on (10) are equivalent to  $Z = \text{Coker}(\phi)$ . Since  $G \otimes S$ - preserves cokernels, we conclude that  $G \otimes_S Z = \text{Coker}(G \otimes \phi)$ , i.e. (11) is semi – exact and  $G \otimes \phi$  is normal.

(3) This follows directly form (2) and the hypothesis on  $G \otimes S$ -. ■

**Lemma 2.3.4.** *Let  $R, S$  be a semirings  ${}_R A_S$  be a  $(R, S)$ -bisemimodule and consider that the functor  $Hom_R(-, A) : {}_R \mathbf{SM} \longrightarrow \mathbf{SM}_S$ . Suppose that*

$$K \xrightarrow{h} L \xrightarrow{f} M \longrightarrow 0 \quad (12)$$

*is a sequence of left  $R$ -semimodules, and assume that the following sequence is a sequence of right  $S$ -semimodules*

$$0 \longrightarrow Hom_R(M, A) \xrightarrow{(f, A)} Hom_R(L, A) \xrightarrow{(h, A)} Hom_R(K, A). \quad (13)$$

(1) *If the sequence  $L \xrightarrow{f} M \longrightarrow 0$  is exact sequence and  $f$  is normal morphism, then the sequence  $0 \longrightarrow Hom_R(M, A) \xrightarrow{(f, A)} Hom_R(L, A)$  is exact sequence and  $(f, A)$  is normal morphism.*

(2) *if the sequence (12) is semi-exact sequence and  $f$  is normal morphism, then the sequence (13) is semi-exact (proper-exact) sequence and  $(f, A)$  is normal morphism.*

(3) *If the sequence (12) is exact sequence and  $Hom_R(-, A)$  converts  $i$ -normal morphisms into  $k$ -normal ones, then the sequence (13) is exact sequence.*

*Proof.* (1) Note that The following are clear: let  $L \xrightarrow{f} M \longrightarrow 0$  is exact sequence, implies that  $f$  is surjective, implies that  $(f, A)$  is injective, implies that  $0 \longrightarrow Hom_R(M, A) \xrightarrow{(f, A)} Hom_R(L, A)$  is exact sequence. Suppose that  $f$  is normal morphism and assume that the exact sequence of  $S$ -semimodules

$$0 \longrightarrow Ker(f) \xrightarrow{i} L \xrightarrow{f} L \longrightarrow 0.$$

Now note that  $M \simeq Coker(i)$ . Now *By Corollary 2.3.1*,  $Hom_R(-, A)$  converts co-kernels into kernels, now we conclude that  $(f, A) = ker((h, A))$  whence normal.

(2) Now *Apply Lemma 2.2.3(5)*:  $K \xrightarrow{h} L \xrightarrow{f} M \longrightarrow 0$  is semi-exact sequence and  $f$  is normal morphism, implies that  $M \simeq Coker(h)$ . Now

because of that the contravariant functor  $Hom_R(-, A)$  converts *cokernels into kernels*, it follows that  $Hom_R(M, A) = Ker((h, A))$  which is in turn equivalent to (13) being semi-exact sequence and  $(f, A)$  being normal. Notice that

$$(f, A)(Hom_S(M, A)) = \overline{(f, A)(Hom_S(M, A))} = Ker((h, A)),$$

i.e. (13) is *proper-exact sequence* (whence *semi-exact sequence*).

(3) This follows directly from (2) and the hypothesis on  $Hom_R(-, A)$ . ■

## Chapter 3

# Projective, Inejective and Flat Semimodules

During this section,  $(S, +, 0, \cdot, 1)$  is a semiring.

### 3.1 Projective Semimodules

There are many notations of projective semimodules in the previous study, but in this Chapter, we study the some properties of projective semimodules and relate them with principal left ideals, and study some properties of projective semimodules, as well as special type of projective semimodules such that *e-projective semimodules* which appeared first in [3].

**Definition 3.1.1.** *Let  $B$ ,  $M$ , and  $N$  are left  $S$ -semimodules, then  $B$  is **projective** if and only if satisfies the following condition: if  $\varphi : N \rightarrow M$  is a surjective  $S$ -homomorphism and if  $\beta : B \rightarrow M$  is an  $S$ -homomorphism, then there exists an  $S$ -homomorphism  $\gamma : B \rightarrow N$  such that  $\varphi \circ \gamma = \beta$ .*

**Definition 3.1.2.** Suppose that  $B$  is a left  $S$ -semimodule, then  $B$  is called

(1) **N-projective** [13] if for any surjective  $S$ -homomorphism  $h : N \rightarrow M$ , and  $f$  is an  $S$ -homomorphism  $f : B \rightarrow M$ , there exists an  $S$ -homomorphism  $g : B \rightarrow N$  such that  $h \circ g = f$ ,

$$\begin{array}{ccccc} N & \xrightarrow{h} & M & \longrightarrow & 0 \\ & \swarrow \exists g & \uparrow f & & \\ & & B & & \end{array}$$

(2)  $N - k$ -**projective**[9] if for any normal epimorphism  $h : N \rightarrow M$  and any  $S$ -homomorphism  $f : B \rightarrow M$ , then there exists an  $S$ -homomorphism  $g : B \rightarrow N$  such that  $h \circ g = f$ .

$$\begin{array}{ccccc} N & \xrightarrow{h(\text{normal})} & M & \longrightarrow & 0 \\ & \swarrow \exists g & \uparrow f & & \\ & & B & & \end{array}$$

(3) **Normally  $N$ -projective**[4] if for any normal epimorphism  $h : N \rightarrow M$  and any  $S$ -homomorphism  $f : B \rightarrow M$ , then there exists an  $S$ -homomorphism  $g : B \rightarrow N$  such that  $h \circ g = f$  and whenever an  $S$ -homomorphism  $h' : B \rightarrow N$  such that  $h \circ h' = f$ , then there exists  $S$ -homomorphisms  $h_1, h_2 : B \rightarrow N$  such that  $f \circ h_1 = 0 = f \circ h_2$  and  $h + h_1 = h' + h_2$ .

$$\begin{array}{ccccccc}
& & h_2 & & h(\text{normal}) & & \\
B & \xrightarrow{\quad} & N & \xrightarrow{\quad} & M & \longrightarrow & 0 \\
& & h_1 & & \exists g & & \\
& & & \swarrow & \uparrow f & & \\
& & & & B & & \\
& & & \searrow h' & & & 
\end{array}$$

Now  $B$  is called that (**resp., k-projective, normally projective**) if  $B$  is (*resp.,  $N$ - $k$ -projective, normally  $N$ -projective*) for every left  $S$ -semimodule  $N$ .

**Proposition 3.1.1.** *Every free left  $S$ -semimodule is projective.*

*Proof.* Suppose  $M, N$  are left  $S$ -semimodules, and  $B$  is a free left  $S$ -semimodule with its basis  $C$ . Now let  $\varphi : N \rightarrow M$  be a surjective  $S$ -homomorphism, and let  $\beta : B \rightarrow M$  be an  $S$ -homomorphism. Now since  $\varphi$  is surjective, then for each element  $b$  of  $B$  there exists an element  $n_b$  of  $N$  such that  $\varphi(n_b) = \beta(b)$ . Now By proposition 2.1.4, there exists a unique  $S$ -homomorphism  $\gamma : B \rightarrow N$  such that  $\gamma(b) = n_b$ . Then  $\beta \circ \gamma(b) = \beta(n_b) = \beta(b)$  for any  $b$  belongs in  $B$  and so, by the uniqueness part of proposition.2.1.4, we must have  $\beta = \beta \circ \gamma$ . ■

**Definition 3.1.3.** *Let  $K, L$  be left  $S$ -semimodules, then  $K$  is called a **retract** of  $L$  if and only if there exist a surjective  $S$ -homomorphism  $\beta : L \rightarrow K$  and an  $S$ -homomorphism  $\gamma : K \rightarrow L$  such that  $\beta \circ \gamma = id_K$ .*

**Remark 3.1.1.** *Let  $K, L$  be left  $S$ -semimodules, if  $K$  is a direct summand of  $L$ , then  $K$  is a retract of  $L$ . Moreover, if  $K$  is a retract of  $L$  and  $L$  is a retract of  $L'$ , then  $K$  is directly a retract of  $L'$ .*

**Proposition 3.1.2.** *A left  $S$ -semimodule is projective  $S$ -semimodule if and only if it is a retract of a free left  $S$ -semimodule.*



*Proof.* ( $\implies$ ) Suppose that  $B$  is a projective left  $S$ -semimodule, then we note that by *proposition 2.1.3*, there exists a *free*  $S$ -semimodule  $A$  and a surjective  $S$ -homomorphism  $\beta : A \twoheadrightarrow B$ . Now because of that  $B$  is *projective*, there exists an  $S$ -homomorphism  $\alpha : B \rightarrow A$  satisfies the following:  $\beta \circ \alpha$  is the identity map on  $B$ .

( $\impliedby$ ) Suppose that  $B$  is a retract of a *free* left  $S$ -semimodule  $A$  and let  $\beta : A \rightarrow B$  and  $\alpha : B \rightarrow A$  be  $S$ -homomorphisms such that  $\beta$  is surjective and  $\beta \circ \alpha$  is the identity map on  $B$ . Let  $L, M$  are left  $S$ -semimodule, and  $\varphi : L \rightarrow M$  be a *surjective*  $S$ -homomorphism, and let  $\psi : B \rightarrow M$  be an  $S$ -homomorphism. Now since that  $A$  is *projective* (by *proposition 3.1.1*), there exists an  $S$ -homomorphism  $\eta : A \rightarrow L$  such that  $\varphi \circ \eta = \psi \circ \beta$ . So  $\varphi \circ \eta \circ \alpha = \psi \circ \beta \circ \alpha = \psi$ , and so  $\eta \circ \alpha : B \rightarrow L$  is a map having the property to show the projectivity.  $\blacksquare$

**Corollary 3.1.1.** *Let  $A$  be Any retract of a projective left  $S$ -semimodule  $B$ , then  $A$  is projective.*

**Proposition 3.1.3.** *Suppose that the set  $\{B_i\}_{i \in I}$  be an arbitrary family of  $S$ -semimodules. Then the direct sum  $B = \bigoplus_{i \in I} B_i$  is projective if and only if each  $B_i$  is projective.*

*Proof.* ( $\implies$ ) If  $B$  is *projective*, then each  $B_i$  is a retract of  $B$  and hence is *projective* by *Corollary 3.1.1*

( $\impliedby$ ) Let  $f : A \rightarrow L$  be an surjective  $S$ -homomorphism of  $S$ -semimodules. Let  $g : B \rightarrow L$  be a  $S$ -homomorphism. Let  $\pi_i : B \rightarrow B_i$  be the canonical projection and  $q_i : B_i \rightarrow B$  be the canonical injection. Define  $g_i : B_i \rightarrow L$  such that  $g_i = g \circ q_i$  for each  $i \in I$ . Since  $B_i$  is *projective*, there exists an  $S$ -homomorphism  $h_i : B_i \rightarrow A$  such that  $f \circ h_i = g_i$  for each  $i \in I$ . Define  $h : B \rightarrow A$  by  $h = \sum_{i \in I} h_i \circ \pi_i$ .

Then

$$f \circ h = f \left( \sum_{i \in I} h_i \circ \pi_i \right) = \sum_{i \in I} f \circ h_i \circ \pi_i = \sum_{i \in I} g_i \circ \pi_i = \sum_{i \in I} g \circ q_i \circ \pi_i = g \sum_{i \in I} q_i \circ \pi_i = g$$

So  $B$  is *projective*. ■

**Definition 3.1.4.** Let  $X, Y$  be Left  $S$ -semimodules and let  $\phi : X \longrightarrow Y$ , be a surjective  $S$ -homomorphism, then  $\phi$  is called **coessential** if and only if for every surjective  $S$ -homomorphism  $\gamma : Z \longrightarrow A$  such that  $\phi \circ \gamma$  is surjective.

**Definition 3.1.5.** Let  $X$  be left  $S$ -semimodule and  $B$  is projective  $S$ -semimodule. If  $\gamma : B \longrightarrow X$  is coessential  $S$ -homomorphism, then  $B$  is called **projective cover** of  $X$ .

**Remark 3.1.2.** If  $S$  is an additivity idempotent semiring. If every left  $S$ -semimodule of  $S$  has projective cover, then  $S$  will be equal  $\{0\}$ .

*Proof.* By contradiction, suppose that  $S \neq \{0\}$ , and  $B \leq S^{\mathbb{N}}$ , where  $B$  is generated by  $\{a_i, \text{ for } i \in \mathbb{N}\}$ , such that the  $k^{\text{th}}$  component  $a_i^k$  of  $a_i$  is equal 0 where  $k < i$ , and otherwise  $a_i^k = 1$ .

**Definition 3.1.6.** [7] Let  $S$  be a semiring, then it is called a right **PP-semiring** if and only if each principal right ideal of  $S$  is projective.

**Definition 3.1.7.** [7] Let  $s$  belongs to  $S$ , then it is called right **e-cancellative** if and only if there exists an element  $e$  belongs in  $S$  such that  $se = s$  holds and  $sz = sb$  implies  $ez = eb$  for any  $z, b$  belongs in  $S$ .

Note from the above definition that  $e$  is a multiplicative idempotent element of  $S$ , since  $s.e = s.1 \implies e.e = e.1 = e$ .

**Proposition 3.1.4.** [7] *Let  $A$  be an arbitrary right ideal of a semiring  $S$  generated by element  $e$  where  $e$  is a multiplicative idempotent element of  $S$ , then  $A$  is projective.*

*Proof.* Since  $Se$  is a retract of the right  $S$ -semimodule  $S_S$ , but  $S_S$  is free, so  $A$  is projective. ■

**Proposition 3.1.5.** [7] *Let  $B_S$  be cyclic right  $S$ -semimodule, then it is projective if and only if  $B \cong aS$ , where  $a$  is a multiplicative idempotent in  $S$ .*

*Proof.* ( $\implies$ ) Assume that  $B_S$  is a cyclic *projective* right  $S$ -semimodule, then  $B = bS$  is satisfied for some  $b \in B$ . Now let  $h : S \rightarrow B = bS$  be  $S$  – *epimorphism* defined by  $h(1) = b$ . Since  $B$  is *projective*, then there exists an  $S$  – *homomorphism*  $f : B \rightarrow S$  satisfies the following:  $h \circ f = i_b$ . Now put  $f(b) = a \in S$ . Then we have that  $b = h \circ f(b) = h(a) = h(1.a) = h(1).a = ba$ . Now  $b = ba$  implies that  $f(b) = f(ba) = f(b)a$ , implies that  $a = a.a$  proves that  $a$  is a multiplicative idempotent of  $S$ . Moreover,  $f(b) = a$  implies that  $f$  is an  $S$  – *epimorphism* of  $B = bS$  onto  $aS \subseteq S$  defined by  $f(by) = f(b)y = ay$  for all  $y$  belongs to  $S$ . We show that this mapping is also injective and hence an  $S$  – *isomorphism* which implies  $B = bS \cong aS$ . Now for any  $s_1, s_2$  belongs to  $S$ , by  $as_1 = as_2$  in  $S$  it follows that  $b(as_1) = b(as_2)$ , implies that  $(ba)s_1 = (ba)s_2$ , implies that  $bs_1 = bs_2$  in  $B$ . (Note that  $as_1 = as_2$  and  $bs_1 = bs_2$  are in fact equivalent because of  $bs_1 = bs_2$  yields in turn  $f(bs_1) = f(bs_2)$  and thus  $as_1 = as_2$ ). ( $\impliedby$ ) If  $B \cong aS$  is satisfies for a multiplicative idempotent  $a \in S$ , then by *proposition 3.1.4* we conclude that  $aS$  is *projective* and hence also  $B$ . ■

**Corollary 3.1.2.** [7] *Let  $S$  be a semiring, and  $b \in S$ , then a principal right ideal  $bS$  is a projective right  $S$ -semimodule if and only if  $b$  is  $e$ -cancellative for some  $e \in S$ .*

**Corollary 3.1.3.** [7] *A semiring  $S$  is right **PP-semiring** if for each principal right ideal of  $S$  is generated by a left  $e$ -cancellative element for some  $e \in S$ .*

**Proposition 3.1.6.** [21] *Let  $K$  be a right  $S$ -semimodule and  $h \in \text{End}_S(K)$ . If  $h(K)$  is projective, then  $h$  is left  $gh$ -cancellative where  $g : h(K) \rightarrow K$  is a monomorphism.*

*Proof.* As  $h : K \rightarrow h(K)$  is an epimorphism and  $h(K)$  is projective, so there exists an  $S$ -homomorphism  $g : h(K) \rightarrow K$  such that  $h \circ g = i_{h(K)}$ . Thus we have  $h \circ (g \circ h) = (h \circ g) \circ h = i_{h(K)} \circ h = h$ . Now let  $h \circ f_1 = h \circ f_2$  for some  $f_1, f_2 \in E_S(K)$ . then  $(g \circ h) \circ f_1 = g \circ (h \circ f_1) = g \circ (h \circ f_2) = (g \circ h) \circ f_2$ . Thus  $h$  is left  $gh$ -cancellative. ■

**Theorem 3.1.7.** [7] *A semiring  $S$  is a right PP-semiring if and only if  $\text{End}_S(B)$  is a right PP-semiring for any cyclic projective right  $S$ -semimodule  $B_S$ .*

*Proof.* ( $\implies$ ) Let  $S$  be a PP-semiring,  $B_S = bS$ , where  $b$  belongs to  $B$  a cyclic projective  $S$ -semimodule and  $h \in \text{End}_S(B)$ . Now by proposition 3.1.5 there is an element  $e$  of  $S$  and an  $S$ -isomorphism  $g : P = pS \rightarrow eS \in S$ . Hence  $g$  maps the (cyclic)  $S$ -subsemimodule  $h(B)$  of  $B$  isomorphically onto a principal right ideal of  $S$ . Now since  $S$  is PP, the latter is projective and thus also  $f(B)$ . We conclude by proposition 3.1.6 that even each element  $h$  of the semiring  $\text{End}_S(B)$  is left  $gh$ -cancellative. Thus, by corollary 3.1.3 that  $\text{End}_S(B)$  is a PP-semiring .

( $\impliedby$ ) Since  $S = 1.S$  is a cyclic and projective right  $S$ -semimodule,  $\text{End}_S(S_S)$  is PP by assumption. Now since  $\text{End}_S(S_S)$  and  $S$  are

isomorphic semirings, then  $S$  is also right  $PP$ . ■

**Definition 3.1.8.** [3] *Let  $N$ ,  $K$ , and  $L$  be  $S$ -semimodules, then  $B$  is called  **$N$ - $e$ -projective** if and only if the covariant functor*

$$\text{Hom}_S(B, -) : {}_S\mathbf{SM} \longrightarrow {}_{\mathbb{Z}^+}\mathbf{SM}$$

*turns any short exact sequence of left  $S$ -semimodules*

$$0 \rightarrow L \xrightarrow{f} N \xrightarrow{g} \rightarrow 0$$

*into a short exact sequence of commutative monoids*

$$0 \longrightarrow \text{Hom}_S(B, K) \xrightarrow{(B,h)} \text{Hom}_S(B, N) \xrightarrow{(B,f)} \text{Hom}_S(B, L) \longrightarrow 0.$$

$B$  is called  **$e$ -projective** if  $B$  is  $N$ - $e$ -projective for any left  $S$ -semimodule  $N$ .

**Remark 3.1.3.** *Every projective and  $e$ -projective semimodules are  $k$ -projective.*

**Proposition 3.1.8.** *Let  $B$  be a left  $S$ -semimodule.*

- ${}_S B$  Is  $N$ - $e$ -projective (for some left  $S$ -semimodule  $N$ ) if and only if  ${}_S B$  is normally  $N$ -projective.
- ${}_S B$  Is  $e$ -projective if and only if  ${}_S B$  is normally projective.

*Proof.* We need to show the first part only:

( $\implies$ ) Assume that  ${}_S B$  is  $N$ - $e$ -projective. Let  $h : N \rightarrow K$  be a

normal epimorphism and  $f : B \rightarrow K$  an  $S$ -homomorphism. Now by lemma 2.2.3 the following sequence is a short exact sequence

$$0 \longrightarrow \text{Ker}(h) \xrightarrow{i} N \xrightarrow{h} K \longrightarrow 0$$

where the map  $i$  is the injection map. Now we have By assumption, the following sequence of commutative monoids is exact:

$$0 \longrightarrow \text{Hom}_S(B, \text{Ker}(h)) \xrightarrow{(B,i)} \text{Hom}_S(B, N) \xrightarrow{(B,f)} \text{Hom}_S(B, K) \longrightarrow 0$$

In particular we note that  $(B, f)$  is surjective and  $k$ -normal, whence  $B$  is  $N$ -projective.

( $\Leftarrow$ ) Let  $A, N$  and  $K$  be left  $S$ -semimodules, and let the sequence  $0 \longrightarrow A \xrightarrow{h} N \xrightarrow{f} K \longrightarrow 0$  be a short exact sequence and consider the induces sequences of commutative monoids

$$0 \longrightarrow \text{Hom}_S(B, A) \xrightarrow{(B,h)} \text{Hom}_S(B, N) \xrightarrow{(B,f)} \text{Hom}_S(B, K) \longrightarrow 0.$$

By proposition 2.3.2  $(B, h)$  we conclude that it is a normal monomorphism and  $\text{Im}((B, h)) = \text{Ker}((B, f))$ . Note that by assumption,  $(B, f)$  is a normal epimorphism, where the induced sequence of commutative monoids is exact. ■

**Proposition 3.1.9.** *Every projective left  $S$ -semimodule is  $e$ -projective.*

*Proof.* Let  $N, K$  be left  $S$ -semimodules, and  ${}_S B$  be projective. Suppose that  $N \xrightarrow{f} K \longrightarrow 0$  a normal epimorphism, and  $\beta \in \text{Hom}_S(B, K)$ . Now since  ${}_S B$  is  $N$ -projective,

$$\text{Hom}_S(B, N) \xrightarrow{(B,f)} \text{Hom}_S(B, K) \longrightarrow 0$$

is surjective, i.e. there exists  $\gamma \in \text{Hom}_S(B, N)$  such that  $f \circ \gamma = \beta$ . Now By proposition 3.1.8, we note that it is enough to show that  $(B, f)$  is  $k$ -normal.

Suppose that  $(B, f)(\gamma) = (B, f)(\gamma')$  for some  $\gamma, \gamma' \in \text{Hom}_S(B, N)$ , i.e.  $f \circ \gamma = f \circ \gamma'$ . Since  ${}_S B$  is projective,  $B$  is a *retract* of a *free* left  $S$ -semimodule, i.e. There exists an index set  $\Lambda$  and a surjective  $S$ -homomorphism  $\eta : S^{(\Lambda)} \rightarrow B$  as well as an injective  $S$ -homomorphism  $\phi : B \rightarrow S^{(\Lambda)}$  such that  $\eta \circ \phi = \text{id}_B$ . Note that  $f \circ \gamma \circ \eta = f \circ \gamma' \circ \eta$  for any  $\lambda \in \Lambda$ , since  $f$  is  $k$ -normal, then there exist  $n_\lambda, n'_\lambda \in \text{Ker}(f)$  such that  $(\gamma \circ \eta)(\lambda) + n_\lambda = (\gamma' \circ \eta)(\lambda) + n'_\lambda$ , let  $\psi, \psi' \in \text{Hom}_S(S^{(\Lambda)}, N)$  be the unique  $S$ -homomorphisms with  $\psi(\lambda) = n_\lambda$  and  $\psi'(\lambda) = n'_\lambda$ , for any  $\lambda \in \Lambda$  (they exist and are unique since  $\Lambda$  is a basis for  $S^{(\Lambda)}$ ). We have

$$f \circ (\psi \circ \phi) = (f \circ \psi) \circ \phi = 0 = (f \circ \psi') \circ \phi = f \circ (\psi' \circ \phi),$$

i.e.  $\psi \circ \phi, \psi' \circ \phi \in \text{Ker}((B, f))$ . Moreover, for any  $\lambda \in \Lambda$  we have that

$$(\gamma \circ \eta + \psi)(\lambda) = (\gamma \circ \eta)(\lambda) + n_\lambda = (\gamma' \circ \eta)(\lambda) + n'_\lambda = (\gamma' \circ \eta + \psi')(\lambda),$$

whence  $\gamma \circ \eta + \psi = \gamma' \circ \eta + \psi'$ . It follows that

$$\begin{aligned} \gamma + \psi \circ \phi &= \gamma \circ \text{id}_B + \psi \circ \phi = \gamma \circ (\eta \circ \gamma) + \phi \circ \phi \\ &= (\gamma \circ \eta + \psi) \circ \phi = (\gamma' \circ \eta + \psi') \circ \phi \\ &= \gamma' \circ (\eta \circ \psi) + \psi' \circ \phi = \gamma' \circ \text{id}_B + \psi' \circ \phi \\ &= \gamma' + \psi' \circ \phi. \end{aligned}$$

■

The following example show that the converse of *proposition 3.1.9* is not true.

**Example 3.1.1.** Let  $\mathbb{Q}^+$  be a *semiring*, where it is the set of non-negative rational numbers, with normal addition and multiplication. Consider that the **Boolean algebra**  $\mathbb{B}$  as a left  $S$ -semimodule (with  $s \cdot 1 = 1$ ) if and only if  $s$  belongs to  $S \setminus \{0\}$ , then we have  ${}_S \mathbb{B}$  is *S-e-projective* but not an *S-projective S-semimodule*.

*Proof.* Consider the  $S$ -homomorphism

$$h : S \rightarrow \mathbb{B}, s \mapsto \{1 \text{ if } s \neq 0, 0 \text{ if } s = 0\}$$

Note that  $h$  is not  $k$ -normal :  $Ker(h) = \{0\}$ ,  $h(1) = 1 = h(2)$ , and  $1+0 \neq 2+0$ . Since there is no surjective  $S$ -homomorphism from  $\mathbb{B}$  to  $S$ , so there is no isomorphism from  $\mathbb{B}$  to  $S$ . Now because of that  $S$  is an *ideal-simple*  $S$ -semimodules, and by Lemma 2.1.2., then  $Hom_S(B, S) = \{0\}$ . Note that the following diagram cannot be commutative:

$$\begin{array}{ccc} S & \xrightarrow{f} & \mathbb{B} \\ & \searrow 0 & \uparrow id_{\mathbb{B}} \\ & & \mathbb{B} \end{array}$$

we conclude  $B$  is not  $S$ -projective. Suppose that  $K$  is an  $S$ -semimodule and  $h : S \rightarrow K$  is a normal  $S$ -epimorphism. If  $h = 0$ , then  $K = h(S) = 0$ , which implies that for any  $S$ -homomorphism  $\beta : \mathbb{B} \rightarrow K$  is the zero morphism and by choosing  $S$ -homomorphism  $0 = \phi : \mathbb{B} \rightarrow S$ , then  $h = \phi \circ \beta$ . If  $h \neq 0$ , then  $h(1) \neq 0$ . Now for any  $a \in S \setminus 0$ , we have  $0 \neq h(1) = h(a^{-1}a) = a^{-1}h(a)$ , whence  $h(a) \neq 0$ . Thus  $Ker(h) = \{0\}$ . If  $h(a) = h(b)$ , then  $a + c_1 = b + c_2$  for some  $c_1, c_2 \in Ker(h) = \{0\}$ , thus  $a = b$ . Hence,  $h$  is an  $S$ -homomorphism and  $K$  is  $S$ -isomorphic to  $S$ . Now since  $S$  is not  $S$ -isomorphic to  $\mathbb{B}$ , then  $K$  is not  $S$ -isomorphic to  $\mathbb{B}$ . Since  $S$  is *ideal-simple*,  $K$  is *ideal-simple*. So  $Hom_S(\mathbb{B}, K) = \{0\}$  and  $\mathbb{B}$  is  $S$ -e-projective. ■

**Definition 3.1.9.** Let  $K, L$  and  $A$  be  $S$ -semimodules, then the following sequence

$$0 \rightarrow K \xrightarrow{h} L \xrightarrow{\phi} A \rightarrow 0 \quad (14)$$

is



- *Left splitting* if there exists  $h' \in \text{Hom}_S(L, K)$  such that  $h' \circ h = \text{id}_K$ .
- *Right splitting* if there exists  $\phi' \in \text{Hom}_S(A, L)$  such that  $\phi \circ \phi' = \text{id}_A$ .

The sequence (14) splits or is *splitting* if it is both (left and right *splitting*) .

**Remark 3.1.4.** *A short exact sequence of modules over a ring is left splitting if and only if it is right splitting. However, this is not the case for semimodules over arbitrary semiring (the following example show that).*

**Example 3.1.2.** *Consider The following sequence:*

$$0 \longrightarrow \{0, 2\} \xrightarrow{i} B(3, 1) \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0, \quad (15)$$

is a short exact sequence of commutative monoids, where  $i$  is the injection map and  $\pi$  is the projection map. Now since  $\{0, 2\}$  is subtractive and  $B(3, 1)/\{0, 2\} \cong_{\mathbb{Z}^+} \mathbb{Z}_2$  (see lemma 2.2.3), we conclude that (15) is exact sequence. Now Consider the following:

$$\phi : B(3, 1) \longrightarrow \{0, 2\}, x \mapsto \{ 2 \text{ if } x \neq 0, \ 0 \text{ if } x = 0 \}$$

and note that  $\phi \circ i = \text{id}_{\{0, 2\}}$ , hence (15) is left splitting semimodule. But, we have  $\text{Hom}_{\mathbb{Z}^+}(\mathbb{Z}_2, B(3, 1)) = \{0\}$  since 1 has an additive inverse (namely 1) in  $\mathbb{Z}_2$ , while no non-zero element of  $B(3, 1)$  has an additive inverse. So, the sequence (15) is not right splitting.

**Proposition 3.1.10.** *Let  $K$ ,  $E$  and  ${}_S B$  be left  $S$ -semimodules, then  ${}_S B$  is  $k$ -projective if and only if every short exact sequence*

$$0 \rightarrow K \xrightarrow{\phi} E \xrightarrow{\alpha} B \rightarrow 0$$

*is right-splitting.*

*Proof.* ( $\implies$ ) Let  $B$  be  $k$ -projective and  $0 \rightarrow Z \xrightarrow{\phi} G \xrightarrow{\alpha} B \rightarrow 0$  be a short exact sequence. In particular  $\alpha$  is surjective and  $k$ -normal. Consider  $id_B : B \rightarrow B$ . Since  ${}_S B$  is  $k$ -projective, then there exists an  $S$ -homomorphism  $\alpha' : B \rightarrow G$  such that the following diagram

$$\begin{array}{ccc} & B & \\ \exists \alpha' \swarrow & \downarrow id_B & \\ G & \xrightarrow{\alpha} & B \end{array}$$

is commutative, i.e.  $\alpha \circ \alpha' = id_B$ .

( $\impliedby$ ) Let  $G \xrightarrow{\alpha} V \rightarrow 0$  be a normal surjective  $S$ -homomorphism and  $\gamma : B \rightarrow V$  be a morphism of left  $S$ -semimodules. Consider the pullback of  $g$  and  $h$ :

$$D = \{(b, g) \in B \times G \mid \gamma(b) = \alpha(g)\}$$

and the following diagram is commutative:

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ \pi_G \downarrow & & \downarrow \gamma \\ G & \xrightarrow{\alpha} & V \end{array}$$

where  $\pi_B$  and  $\pi_D$  are the projection maps. Since  $\alpha$  is surjective,  $\gamma(b) = \alpha(g)$  for some  $g \in G$ , i.e.  $(b, g) \in D$  and indeed,  $b = \pi_B(b, g)$ . So,  $\pi_B$  is surjective. Let  $(p, g), (b, g') \in D$  so,  $\pi_B(b, g) = \pi_B(b, g')$ . Then,  $\alpha(g) = \gamma(b) = \alpha(g')$  and there exist  $k, k' \in Ker(\alpha)$  such that  $g + k = g' + k'$  (since  $\alpha$  is  $k$ -normal). Note  $(0, k), (0, k') \in Ker(\pi_B)$  and  $(b, g) + (0, k) = (b, g + k) = (b, g' + k') = (b, g) + (0 + k')$ , i.e.  $\pi_B$

is  $k$  – normal. Hence the sequence

$$0 \rightarrow Ker(\pi_B) \hookrightarrow D \xrightarrow{\pi_B} B \rightarrow 0$$

is exact, and by our hypothesis there exists an  $S$ -homomorphism  $\psi : B \rightarrow D$  such that  $\pi_B \circ \psi = id_B$ . Note for any  $b \in B$ ,  $\psi(b) \in D$  whence  $\psi(b) = (b, g)$  for some  $g$  belongs to  $G$  with  $\gamma(b) = \alpha(g)$ . It follows that

$$(\alpha \circ (\pi_G \circ \psi))(b) = \alpha(\pi_G(b, g)) = \alpha(g) = \gamma(b). \quad (16)$$

So,  $\alpha \circ (\pi_G \circ \psi) = \gamma$ . So,  $B$  is  $k$ -projective. ■

**Lemma 3.1.11.** *Let  $N$  be a left  $S$ -semimodule such that every subtractive subsemimodule is a direct summand, then every left  $S$ -semimodule is  $N$ -e-projective.*

*Proof.* Suppose that  $B$  is a left  $S$ -semimodule and let

$$\phi : N \longrightarrow A \longrightarrow 0$$

be a normal epimorphism and  $\alpha : B \rightarrow A$  be an  $S$ -homomorphism. Note  $Ker(\phi) \leq N$  is a subtractive subsemimodule, whence  $N = Ker(\phi) \oplus Y$ , where  $Y \leq N$ . The row of this following diagram is exact

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ker(\phi) & \xrightarrow{i} & N & \xrightarrow{\phi} & A \longrightarrow 0 \\ & & & & & & \uparrow \alpha \\ & & & & & & B \end{array}$$

by lemma 2.2.3 and it follows (see also remarks 2.1.4) that isomorphisms of left  $S$ -semimodules:

$$A \simeq N/Ker(\phi) \simeq Y.$$

Consider the following isomorphism  $A \xrightarrow{\alpha'} Y$  and setting  $\gamma := i_Y \circ \alpha' \circ \alpha : B \rightarrow N$  where  $\phi \circ i_Y = id_Y$  and  $i_Y \circ \phi|_Y = id_A$ ,  $\phi \circ \gamma = \alpha$ . Assume that also  $\gamma' : B \rightarrow N$  such that  $\phi \circ \gamma' = \alpha$ . Consider the projection  $\beta : N \rightarrow Ker(\phi)$ . Then

$$\eta : N \rightarrow N, n \mapsto i_Y \circ \alpha' \circ \phi + \beta$$

is the identity map. Let  $n \in N$ , and write  $n = a + y$  for some unique  $a$  belong to  $Ker(\phi)$  and  $y$  belongs in  $Y$ , and note

$$\eta(n) = \eta(a+y) = (i_Y \circ \alpha' \circ \phi + \beta)(a+y) + (i_Y \circ \alpha' \circ \phi)(a+y) + \beta(a+y) = y+a = n.$$

Choose  $\gamma_1 := \beta \circ \gamma' : B \rightarrow N$  and  $\gamma_2 = 0 : B \rightarrow N$ . Note that  $\phi \circ \gamma_1 = \phi \circ \beta \circ \gamma' = 0 = \phi \circ \gamma_2$ . Moreover, we have for each  $b \in B$ :

$$\begin{aligned} (\gamma + \gamma_1)(b) &= \gamma(b) + \gamma_1(b) = (i_Y \circ \alpha' \circ \alpha)(b) + (\beta \circ \gamma')(b) \\ &= (i_Y \circ \alpha' \circ \phi \circ \gamma')(b) + \beta \circ \gamma'(b) = ((i_Y \circ \alpha' \circ \phi + \beta) \circ \gamma')(b) \\ &= \gamma'(b) = (\gamma' + 0)(b). \end{aligned}$$

Therefore,  $B$  is  $N - e - projective$ . ■

**Lemma 3.1.12.** [3]

- *Let  $N$  be a left  $S$ -semimodule, then a retract of an  $N$ - $e$ -projective semimodule is  $N$ - $e$ -projective.*
- *A retract of an  $e$ -projective left  $S$ -semimodule is  $e$ -projective.*

*Proof.* We need to prove only the first part: Let  $N$  and  $B$  be left  $S$ -semimodules, which  $B$  is  $N - e - projective$  and let  ${}_S A$  be a retract of  $B$  along with a surjective  $S$ -homomorphism  $\phi_A : B \rightarrow A$  and an

injective  $S$ -homomorphism  $i_A : A \rightarrow B$  such that  $\pi_A \circ i_A = id_A$ .  
Let  $\phi : N \rightarrow C$  be a normal epimorphism and  $\beta : A \rightarrow C$  an  $S$ -homomorphism. Since  $B$  is e-projective, there exists an  $S$ -homomorphism  $\theta^* : B \rightarrow N$  such that  $\phi \circ \theta^* = \beta \circ \pi_A$ . Consider  $\theta := \theta^* \circ i_A : A \rightarrow A$ .

$$\begin{array}{ccccc}
N & \xrightarrow{\phi} & C & \longrightarrow & 0 \\
\uparrow & & \uparrow \beta & & \\
A & & & & \\
\downarrow i_A & & \downarrow \pi_A & & \\
B & & & & 
\end{array}
\quad \begin{array}{l} \\ \\ \\ \exists \theta^* \end{array}$$

Then  $\phi \circ \theta = \phi \circ (\theta^* \circ i_A) = \beta \circ \pi_A \circ i_A = \beta \circ id_A = \beta$ .  
Assume that  $\theta' : A \rightarrow N$  is an  $S$ -homomorphism such that  $\phi \circ \theta' = \beta$ .  
Since  $B$  is  $N$ -e-projective and  $\phi \circ (\theta' \circ \pi_A) = (\phi \circ \theta') \circ \pi_A = \beta \circ \pi_A$ ,  
there exist  $S$ -homomorphisms

$\theta'_1, \theta'_2 : B \rightarrow N$  such that  $\phi \circ \theta'_1 = 0 = \phi \circ \theta'_2$  and  $\theta^* + \theta'_1 = \theta' \circ \pi_A + \theta'_2$ .  
Consider  $\theta_1 := \theta'_1 \circ i_A$  and  $\theta_2 := \theta'_2 \circ i_A$ .

$$\begin{array}{ccccccc}
A & \xrightarrow{i_A} & B & \xrightarrow{\exists \theta'_2} & N & \xrightarrow{\phi} & C \longrightarrow 0 \\
& & & & \uparrow \theta & & \uparrow \beta \\
& & & & A & & \\
& & & & \downarrow \theta' & & \downarrow \pi_A \\
& & & & B & & 
\end{array}
\quad \begin{array}{l} \\ \\ \\ \exists \theta'_1 \\ \theta^* \end{array}$$

Then  $\phi \circ \theta_1 = \phi \circ \theta'_1 \circ i_A = 0$ ,  $\phi \circ \theta_2 = \phi \circ \theta'_2 \circ i_A = 0$ , and

$$\begin{aligned} \theta + \theta_1 &= \theta^* \circ i_A + \theta'_1 \circ i_A = (\theta^* + \theta'_1) \circ i_A \\ &= (\theta' \circ \pi_A + \theta'_2) \circ i_A = \theta' \circ \pi_A \circ i_A + \theta'_2 \circ i_A \\ &= \theta' + \theta_2. \end{aligned}$$

Consequently,  $A$  is  $N$ - $e$ -projective. ■

**Proposition 3.1.13.** *Let the set  $\{B_i\}_{i \in I}$  be a family of left  $S$ -semimodules, where  $I$  is the index set and  $N$  a left  $S$ -semimodule, then  $\bigoplus_{i \in I} B_i$  is  $N$ - $e$ -projective if and only if  $B_i$  is  $N$ - $e$ -projective for any  $i \in I$ . The class of  $e$ -projective left  $S$ -semimodules is closed under direct sums.*

*Proof.* ( $\implies$ ) Let  $\bigoplus_{i \in I} B_i$  is  $N$ - $e$ -projective  $S$ -semimodule, and note it is retract to each  $\{B_i\}$ , for each  $i \in I$ , and so we have  $\{B_i\}$  is  $N$ - $e$ -projective  $S$ -semimodule for each  $i \in I$  (by Lemma 3.1.12). ( $\impliedby$ ) Let  $\beta : N \rightarrow A$  be a normal epimorphism and  $\phi : \bigoplus_{i \in I} B_i \rightarrow A$  be an  $S$ -homomorphism. For any  $k \in I$ , there exist an  $S$ -homomorphism  $\gamma_k : B_k \rightarrow N$  such that  $\phi \circ i_k = \beta \circ \gamma_k$ , where  $i_k : B_k \rightarrow \bigoplus_{i \in I} B_i$  is the canonical injection.

$$\begin{array}{ccccc} N & \xrightarrow{\beta} & A & \longrightarrow & 0 \\ & \nearrow \exists \gamma & \uparrow \phi & & \\ & & \bigoplus_{i \in I} B_i & & \\ & \nearrow \exists \gamma_k & \uparrow i_k & & \\ & & B_k & & \end{array}$$

By the Universal Property of Direct Coproducts, there exists a unique  $S$ -homomorphism  $\gamma : \bigoplus_{i \in I} B_i \rightarrow N$  such that  $\gamma \circ i_k = \gamma_k$  for any  $k \in I$ , i.e.

$$\gamma : \bigoplus_{i \in I} B_i \rightarrow N, \sum_{i \in I} b_i \mapsto \sum_{i \in I} \gamma_i(b_i).$$

Note  $\gamma$  is well defined function Since the  $\sum_{i \in I} B_i$  is finite (all but finitely many of the coordinates are zero, and it is  $S$ -homomorphism function. And we have

$$\begin{aligned} (\beta \circ \gamma)\left(\sum_{i \in I} b_i\right) &= \beta\left(\sum_{i \in I} \gamma_i(b_i)\right) = \sum_{i \in I} (\beta \circ \gamma_i)(b_i) \\ &= \sum_{i \in I} (\phi \circ i_i)(b_i) = \phi\left(\sum_{i \in I} i_i(b_i)\right) \\ &= \phi\left(\sum_{i \in I} b_i\right). \end{aligned}$$

Now assume that  $\gamma' : \bigoplus_{i \in I} B_i \rightarrow N$  is an  $S$ -homomorphism such that  $\beta \circ \gamma' = \phi$ , then  $\phi \circ i_k = \beta \circ \gamma' i_k$  for any  $j \in I$ . Since  $B_j$  is e-projective for any  $k \in I$ , there exist an  $S$ -homomorphisms  $\tilde{\gamma}_k, \hat{\gamma}_k : B_j \rightarrow N$  such that  $\beta \circ \tilde{\gamma}_k = 0 = \beta \circ \hat{\gamma}_k$  and  $\gamma_k + \tilde{\gamma}_k = \gamma'_k + \hat{\gamma}_k$ .  
Since the Universal Property of Direct Co-products, there exists  $S$ -homomorphisms

$$\tilde{\gamma}, \hat{\gamma} : \bigoplus_{i \in I} B_i \rightarrow N, \tilde{\gamma}\left(\sum_{i \in I} b_i\right) := \sum_{i \in I} \tilde{\gamma}_i(b_i) \text{ and } \hat{\gamma}\left(\sum_{i \in I} b_i\right) = \sum_{i \in I} \hat{\gamma}_i(b_i)$$

$$\begin{array}{ccccccc}
& & \exists \hat{\gamma} & & & & \\
& & \dashrightarrow & & \beta & & \\
B & & N & \xrightarrow{\quad} & A & \longrightarrow & 0 \\
& & \exists \tilde{\gamma} & & & & \\
& & \uparrow & \swarrow \gamma & \uparrow \phi & & \\
& & & & \bigoplus_{i \in I} B_i & & \\
& & \uparrow \gamma' & & \uparrow i_k & & \\
& & & & B_k & & \\
& & \uparrow \gamma_k & & & & 
\end{array}$$

Note that both  $\hat{\gamma}$ ,  $\tilde{\gamma}$  are  $S$ -homomorphism well defined, since the sum  $\sum_{i \in I} b_i$  is finite (all but finitely many of the coordinates are zero).

And we have

$$(\beta \circ \tilde{\gamma})\left(\sum_{i \in I} b_i\right) = \beta\left(\sum_{i \in I} \tilde{\gamma}_i(b_i)\right) = \sum_{i \in I} (\beta \circ \tilde{\gamma}_i)(b_i) = 0$$

$$\hat{\gamma}\left(\sum_{i \in I} b_i\right) = \gamma\left(\sum_{i \in I} \hat{\gamma}_i(b_i)\right) = \sum_{i \in I} (\beta \circ \hat{\gamma}_i)(b_i) = 0$$

and

$$\begin{aligned}
(\gamma + \tilde{\gamma})\left(\sum_{i \in I} b_i\right) &= \gamma\left(\sum_{i \in I} b_i\right) + \tilde{\gamma}\left(\sum_{i \in I} b_i\right) = \sum_{i \in I} \gamma_i(b_i) + \sum_{i \in I} \tilde{\gamma}_i(b_i) \\
&= \sum_{i \in I} (\gamma_i + \tilde{\gamma}_i)(b_i) = \sum_{i \in I} (\gamma'_i + \hat{\gamma}_i)(b_i) \\
&= (\gamma' + \hat{\gamma})\left(\sum_{i \in I} b_i\right).
\end{aligned}$$

Hence  $\bigoplus_{i \in I} B_i$  is  $N$ - $e$ -projective. ■



## Examples

- Let  $S = M_2(\mathbb{R}^+) = N_1 \oplus N_2$ , where

$$N_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{R}^+ \right\} \text{ and } N_2 = \left\{ \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \mid c, d \in \mathbb{R}^+ \right\}$$

and consider

$$L = \left\{ \begin{pmatrix} u & u \\ v & v \end{pmatrix} \mid u, v \in \mathbb{R}^+ \right\}, \text{ and } B = S/L.$$

then

$$0 \rightarrow N_1 \xrightarrow{i_{N_1}} S \xrightarrow{\pi_{N_2}} 0$$

(where  $i$  is the injection map and  $\pi$  is the projection map)

is exact sequence, then  $B$  is  $N_1 - e -$  projective and  $N_2 - e -$  projective and it is also  $B$  is normally  $N_1 -$  projective, and normally  $N_2 -$  projective (By the part one of proposition 3.1.8).

- Let  $\mathbb{Q}^+$  be a semiring, where it is the set of non-negative rational numbers, with normal addition and multiplication. Consider that the **Boolean algebra**  $\mathbb{B}$  as a left  $S$ -semimodule (with  $s \cdot 1 = 1$ ) if and only if  $s$  belongs to  $S/\{0\}$ , then  ${}_s\mathbb{B}$  is  $e -$  projective and it is also normally projective (By the part two of proposition 3.1.8)

## 3.2 Injective Semimodules

During this section, as before  $(S, +, 0, \cdot, 1)$  is a semiring. There are many notations of injective semimodules in the previous study which coincide if it were a module over a ring. but in this section, we study the some properties of injective semimodules and special type of injective semimodules such that  $N$  – *injective*,  $e$  – *injective*,  $N$  –  $e$  – *injective*, *normally injective*[4] and the relation between them. Was first used in [3].

**Definition 3.2.1.** [13] *Let  $S$  be a semiring. A left  $S$ -semimodule  $I$  is called **injective** if and only if for a left  $S$ -semimodule  $A$  and a subsemimodule  $B(B \leq A)$ , any  $S$ -homomorphism from  $\phi : B \longrightarrow I$  can be extended to an  $S$ -homomorphism  $\beta : A \longrightarrow I$ .*

**Definition 3.2.2.** [4] *Let  $N, A$  be left  $S$ -semimodules, then  $A$  is called*

- $N$  –  $e$ -**injective** if the contra variant functor

$$Hom_S(-, A) : {}_S\mathbf{SM} \longrightarrow {}_{\mathbb{Z}^+}\mathbf{SM}$$

turns that any short exact sequence

$0 \longrightarrow X \xrightarrow{\phi} N \xrightarrow{\beta} Y \longrightarrow 0$ , where  $X$  and  $Y$  are left  $S$  – semimodules into a short exact sequence of commutative monoids

$$0 \longrightarrow Hom_S(Y, J) \rightarrow Hom_S(N, A) \rightarrow Hom_S(X, A) \longrightarrow 0.$$

- $e$ -**injective** if  $A$  is  $N$  –  $e$ -injective for any left  $S$ -semimodule  $N$ .

**Definition 3.2.3.** *Let  $A$  and  $N$  be a left  $S$ -semimodule, then  $A$  is called*

- $N$ -**injective**[13] if for any injective  $S$ -homomorphism  $\phi : X \rightarrow N$  and any  $S$ -homomorphism  $\beta : X \rightarrow A$ , then there exists an  $S$ -homomorphism  $\gamma : N \rightarrow A$  such that  $\gamma \circ \phi = \beta$

$$\begin{array}{ccccc}
0 & \longrightarrow & X & \xrightarrow{\phi} & N \\
& & \downarrow \beta & \swarrow \exists \gamma & \\
& & A & & 
\end{array}$$

- ***N-i-injective***[9] if for any normal monomorphism  $\phi : X \rightarrow N$  and every  $S$ -homomorphism  $\gamma : X \rightarrow A$ , then there exists an  $S$ -homomorphism  $\gamma' : N \rightarrow A$  such that  $\gamma' \circ \phi = \gamma$ :

$$\begin{array}{ccccc}
0 & \longrightarrow & X & \xrightarrow{\phi(\text{normal})} & N \\
& & \downarrow \beta & \swarrow \exists \gamma & \\
& & A & & 
\end{array}$$

- **Normally N-injective**[4] if for any normal monomorphism  $\phi : X \rightarrow N$  and any  $S$ -homomorphism  $\beta : X \rightarrow A$ , then there exists an  $S$ -homomorphism  $\gamma : N \rightarrow A$  such that  $\gamma \circ \phi = \beta$ , and whenever an  $S$ -homomorphism  $\gamma' : N \rightarrow A$  such that  $\gamma' \circ \phi = \beta$ , then there exist  $S$ -homomorphisms  $\gamma_1, \gamma_2 : N \rightarrow A$  satisfies that  $\gamma_1 \circ \phi = 0 = \gamma_2 \circ \phi$  and  $\gamma + \gamma_1 = \gamma' + \gamma_2$ :

$$\begin{array}{ccccccc}
0 & \longrightarrow & X & \xrightarrow{\phi(\text{normal})} & N & \overset{\exists \gamma_1}{\dashrightarrow} & A \\
& & \downarrow \beta & \swarrow \exists \gamma & \searrow \exists \gamma_2 & & \\
& & A & & & & 
\end{array}$$

$\gamma'$

Note that  $A$  is **injective semimodule**(resp., ***i*-injective**, **normally injective**) if  $A$  is  $N$ -injective (resp.,  $N - i$ -injective, normally

$N$ -injective) for any left  $S$ -semimodule  $N$ .

**Proposition 3.2.1.** [13] *Let  $S$  be a semiring such that  $S$  is an entire, cancellative, zerosumfree semiring then the only injective left  $S$ -semimodule is  $\{0\}$ .*

*Proof.* Let  $I$  be an injective left  $S$ -semimodule and let  $i$  belong to  $I$ , let  $f_i : S \rightarrow I$  is an  $S$ -homomorphism defined by  $s \mapsto si$ . Since  $I$  is injective, then there exists an  $S$ -homomorphism  $g_i : S^\Delta \rightarrow I$  extending  $f_i$ . Then  $i + g_i(-1) = g_i(1) + g_i(-1) = g_i(0) = g_i(0 \cdot 0) = 0 \cdot g_i(0) = 0$ , and so  $i$  has an additive inverse in  $I$ . So,  $I$  is an  $S$ -module. Let  $I' = I \cup \{\infty\}$  such that defined as the following: ( $I \cup \{\infty\}$  to be the set  $I \cup \{\infty\}$ , where the addition and scalar multiplication from  $I$  have been extended by setting  $i' + \infty = \infty + i' = \infty$  for any  $i'$  belong is  $I \cup \{\infty\}$ ,  $a \cdot \infty = \infty$  for any  $a \in S$  and  $0 \cdot \infty = 0_A$ ). Then the identity map on  $I$  can be extended to an  $S$ -homomorphism  $g : I' \rightarrow I$ . Now Set  $x = g(\infty)$ , for any  $i \in I$  we have  $i + x = g(i) + g(\infty) = g(i + \infty) = g(\infty) = x$ . So we have a contradiction about that  $x$  has an additive inverse any element of  $I$ , has an additive inverse unless  $x = 0$ . So we conclude that  $I = 0$ . ■

**Proposition 3.2.2.** [13] *Let  $S$  be a semiring and let  $I$  be an injective left  $S$ -semimodule. Then*

- $I^X$  is an injective left  $S$ -semimodule for any nonempty set  $X$ .
- Any direct summand of  $I$  is injective.

*Proof.* The proof of The first part: the first thing need to show  $I^X$  is a left  $S$ -semimodule with operations of scalar multiplication and addition defined elementwise as the following: if  $\phi, \gamma \in I^X$  and  $a \in S$ , then  $(\phi + \gamma)(x) = \phi(x) + \gamma(x)$  and  $(a\phi)(x) = a\phi(x)$  for any  $x \in X$ ,

so  $I^X$  is a left  $S$ -semimodule. If  $L$  is an  $S$ -subsemimodule of a left  $S$ -semimodule  $Y$  ( $L \leq Y$ ) and if  $\eta$  is an  $S$ -homomorphism from  $L$  to  $I^X$ , then for any  $x \in X$  there exists an  $S$ -homomorphism  $\eta_x : L \rightarrow I$  defined by  $\eta_x(l) = (\eta(l))(x)$ , where  $l \in L$ . Since  $I$  is injective for every  $x \in X$  there exists an  $S$ -homomorphism  $f_x : Y \rightarrow I$  extending  $\eta_x$ . Define a function  $f : Y \rightarrow I^X$  by  $(f(y))(x) = f_x(y)$  for any  $y \in Y$  and for any  $x \in X$ , then  $f$  is an  $S$ -homomorphism extending  $\eta$ . So  $I^X$  is injective.

The proof of the second part: Let  $I'$  be a direct summand of  $I$  and let  $I''$  be a subsemimodule of  $I$  such that  $I = I' \oplus I''$ , then there exists a surjective  $S$ -homomorphism  $\pi : I \rightarrow I'$ , where its kernel is precisely  $I''$ . Let  $\beta : I' \rightarrow I$  be the inclusion map. If  $L$  is a subsemimodule of a left  $S$ -semimodule  $Y$  ( $L \leq Y$ ) and if  $\eta : L \rightarrow I'$  is an  $S$ -homomorphism then, since  $I$  is injective, there exists an  $S$ -homomorphism  $f : Y \rightarrow I$  extending  $\beta \circ \eta$ . In particular, if  $v \in L$ , then  $f(v) \in I'$  and so  $(\pi \circ f)(v) = (\pi \circ \beta \circ \eta)(v) = \eta(v)$ , so  $\pi \circ \eta : Y \rightarrow I'$  extends  $\eta$ , proving that  $I'$  is injective. ■

**Corollary 3.2.1.** *If  $A$  is a nonempty set the  $\mathbb{B}^A$  is injective as a left  $\mathbb{B}$ -semimodule.*

Let  $S, T$  be semirings, and let  $\phi : S \rightarrow T$  be a semiring homomorphism. If  $T$  is canonically a left  $S$ -semimodule, if we define scalar multiplication by  $s.t = \phi(s).t$  for any  $s \in S$  and  $t \in T$ . Now let  $A$  be a left  $S$ -semimodule. Then  $Hom_S(T, A)$  is a left  $T$ -semimodule with respect to componentwise addition and scalar multiplication given by  $\phi(t') : t \rightarrow \phi(t.t')$  for any  $\phi \in Hom_S(T, A)$  and  $t, t' \in T$ .

**Proposition 3.2.3.** [13] *Let  $S, T$  are semirings, and let  $h : S \rightarrow T$  be a semiring homomorphism. If  $A$  is an injective left  $S$ -semimodule, then  $Hom_S(T, A)$  is injective as a left  $T$ -semimodule.*

*Proof.* Suppose that  $A$  is an injective left  $S$ -semimodule and set  $L = \text{Hom}_S(T, A)$ . Let  $X'$  be a subsemimodule of a left  $T$ -semimodule  $X$  ( $X' \leq X$ ) and let  $g : X' \rightarrow L$  be an  $S$ -homomorphism. Note that  $X$  is a left  $S$ -semimodule with scalar multiplication defined by  $s.x = h(s)x$  for any  $s \in S$  and  $x \in X$ . Moreover,  $X'$  is an  $S$ -subsemimodule of  $X$ . Suppose that  $\varphi : X' \rightarrow A$  is a function defined by  $\varphi : x \mapsto (g(x))(1)$ , then  $\varphi$  is an  $S$ -homomorphism, as can easily be verified. Since  $A$  is injective left  $S$ -semimodule, there exists an  $S$ -homomorphism  $\lambda : X \rightarrow A$  extending  $\varphi$ . Claim that the function  $\eta : X \rightarrow L$  defined by  $\eta(x) : r \mapsto \lambda(tx)$  is an  $R$ -homomorphism. Indeed, for any  $x_1, x_2 \in X$  and for any  $t \in T$  we have

$$\begin{aligned} (t)(\eta(x_1 + x_2)) &= \lambda(t(x_1 + x_2)) = \lambda(tx_1 + tx_2) \\ &= \lambda(tx_1)\lambda(tt_2) = (t)(\eta(x_1)) + (t)(\eta(x_2)) \\ &= (t)(\eta(x_1) + \eta(x_2)) \end{aligned}$$

and for any  $x \in X$  and  $t, t' \in T$  we have  $(t)(\eta(t'x)) = \lambda(t(t'x)) = \lambda((tt')x) = (tt')\eta(x) = (t)(t'\eta(x))$ . This proves the claim. Moreover,  $\eta$  extends  $f$  since for every  $x' \in X'$  and  $t \in T$  we have that  $r(\eta(x')) = \lambda(tx') = \varphi(tx') = (1)(g(tx')) = (1)(tg(x')) = (t)(g(x'))$ .

■

**Definition 3.2.4.** [13] An  $S$ -monomorphism  $\alpha : M \rightarrow N$  of left  $S$ -semimodules is **essential** if and only if for any  $S$ -homomorphism  $\beta : N \rightarrow N'$ , the map  $\beta \circ \alpha$  is an  $S$ -monomorphism only when  $\beta$  is an  $S$ -monomorphism.

**Definition 3.2.5.** Let  $K$  be  $S$ -semimodule and  $K'$  be a subsemimodule of a left  $S$ -semimodule  $K$  ( $K' \leq K$ ), then  $K'$  is **large** in  $K$  if and only if the inclusion map  $K' \rightarrow K$  is an essential  $S$ -homomorphism.

Equivalently, the function  $\alpha : M \rightarrow N$  is an essential  $S$ -homomorphism if and only if  $\alpha(M)$  is a large subsemimodule of  $N$ . It follows that a

subsemimodule  $K'$  of a left  $S$ -semimodule  $K$  is large in  $K$  if and only if any subsemimodule of  $K$  containing  $K'$  is large in  $K$ .

**Proposition 3.2.4.** [13] *Let  $K$  be a  $S$ -semimodule and  $A$  is a subsemimodule of a left  $S$ -semimodule  $A$ , then the following are equivalent:*

- (1)  $A$  is large in  $K$ .
- (2) Suppose that  $\rho$  is a nontrivial  $S$ -congruence relation on  $K$ , then the restriction of  $\rho$  to  $A$  is also nontrivial.
- (3) Suppose that  $k$  and  $k'$  are distinct elements of  $K$ , then there exist distinct elements  $a$  and  $a'$  of  $A$  satisfying  $a \rho(k, k') a'$ .

*Proof.* (1)  $\implies$  (2): Let  $\rho$  be a nontrivial  $S$ -congruence relation on  $K$ , and let  $f : K \rightarrow K/\rho$  be an  $S$ -homomorphism defined by  $k \mapsto k/\rho$ , then  $f$  is not an  $S$ -monomorphism and hence, by (1) neither is its restriction to  $A$ . So there are elements  $a \neq a'$  of  $A$  such that  $a \rho a'$ , so proving that the restriction of  $\rho$  to  $A$  is nontrivial.

(2)  $\implies$  (3): Since  $\rho$  is a nontrivial  $S$ -congruence relation on  $K$ , so we can find distinct elements in  $K$  and  $A$  satisfying  $a \rho(k, k') a'$ .

(3)  $\implies$  (1): Let  $f : K \rightarrow K'$  be an  $S$ -homomorphism, then the restriction of which to  $A$  is an  $S$ -monomorphism. If  $f$  is not injective, then there exist distinct elements  $k$  and  $k'$  of  $K$  such that  $k \rho f(k)$ . By (3), there exist distinct elements  $a$  and  $a'$  of  $A$  such that  $k \rho(k, k') a'$  and so,  $a \rho f(a')$ , which is a contradiction. Thus  $f$  must be an  $S$ -monomorphism, proving(1). ■

**Proposition 3.2.5.** [13] *Let  $I$  be an injective left  $S$ -semimodule, then any essential  $S$ -monomorphism  $f : I \rightarrow I'$  is an  $S$ -isomorphism.*

*Proof.* Let  $I$  be injective and let  $f : I \rightarrow I'$  be an essential  $S$ -monomorphism, then there exists an  $S$ -homomorphism  $g : I' \rightarrow I$  such that  $g \circ f$  is the identity map on  $I$ . Suppose  $a \in I' \setminus f(I)$ , then  $g(a) \in I$  so  $f \circ g(a) \neq a$ . Since  $f \circ g(a) \equiv_g a$ , that means  $\equiv_g$  is a nontrivial  $S$ -congruence relation on  $I'$  therefore, by proposition 3.2.4

$f(v) = g \circ f(v')$ , contradicting the choice of  $g$ . Therefore must have  $I' = f(I)$  and so  $f$  is an  $S$ -isomorphism. ■

**Definition 3.2.6.** [13] *Let  $A$  be a left  $S$ -semimodule. If there exists an injective left  $S$ -semimodule  $I$  and an essential  $S$ -monomorphism  $f : A \rightarrow I$ , then  $I$  is an **injective hull** of  $A$ .*

Note that by *proposition 3.2.* the injective hulls of nonzero  $S$ -semimodules need not exist for every semiring  $S$ .

**Proposition 3.2.6.** [13] *Let  $f : K \rightarrow I$  and  $f' : K \rightarrow I'$  are injective hulls of a left  $S$ -semimodule  $K$ , then there exists an  $S$ -isomorphism from  $I$  to  $I'$ .*

*Proof.* By injectivity, there exists an  $S$ -homomorphism  $\eta : I \rightarrow I'$  such that  $\eta \circ f = f'$ . Claim that this is the isomorphism we seek. Indeed, since  $f' = \eta \circ f$  is an  $S$ -monomorphism, we see by essentiality that  $\eta$  is also an  $S$ -monomorphism. Suppose  $\eta : I' \rightarrow A$  is an  $S$ -homomorphism such that  $g \circ \eta$  is an  $S$ -monomorphism. Then  $g \circ \eta \circ f = g \circ f'$  is also an  $S$ -monomorphism. But,  $f'$  is essential and so  $g$  is an  $S$ -monomorphism. Thus,  $\eta$  is essential and so, we conclude by *proposition 3.2.5*, it is an  $S$ -isomorphism, as claimed. ■

**Proposition 3.2.7.** *Let  $N$  and  $A$  be a left  $S$ -smemodules, then:*

- (1)  $A$   **$N$ -e-injective** if and only if  $A$  is normally  $N$ -injective.
- (2)  ${}_S A$  is **e-injective** if and only if  ${}_S A$  is normally injective.

*Proof.* we need to prove only the first part: let  $N$  be a left  $S$ -semimodule. ( $\implies$ ) Suppose that  $A$  is  $N$ -e-injective. Let  $K \leq N$  be a subtractive  $S$ -subsemimodule. By *lemma 2.2.3*, a short exact sequence of left  $S$ -semimodules

$$0 \longrightarrow K \xrightarrow{i} N \xrightarrow{\pi} N/K \longrightarrow 0$$



where  $i$  is the *injection map* and  $p$  is the *projection map*. By the assumption, the contravariant functor  $Hom_S(-, H) : {}_S\mathbf{SM} \longrightarrow {}_{\mathbb{Z}^+}\mathbf{SM}$  preserves exact sequences, whence the following sequence of commutative monoids

$$0 \longrightarrow Hom_S(N/K, H) \xrightarrow{(\pi, H)} Hom_S(N, H) \xrightarrow{(i, H)} Hom_S(K, H) \longrightarrow 0$$

is exact. especially,  $(i, H) : Hom_S(N, H) \longrightarrow Hom_S(K, H)$  is a *normal epimorphism*, i.e.  $H$  is *normally  $N$ -injective*.

( $\Leftarrow$ ) Let  $K, N$  and  $A$  are left  $S$ -semimodules, and the following sequence:

$$0 \longrightarrow K \xrightarrow{\phi} N \xrightarrow{\beta} A \longrightarrow 0 \quad (17)$$

be an exact sequence. Now applying the contravariant functor  $Hom_S(-, H)$  on (17), so we have that by *lemma 2.3.4(2)* and our assumption that the following sequence of *commutative monoids*:

$$0 \longrightarrow Hom_S(K, H) \xrightarrow{(\beta, H)} Hom_S(N, H) \xrightarrow{(\phi, H)} Hom_S(A, H) \longrightarrow 0$$

is exact, i.e.  ${}_S\mathbf{H}$  is exact. ■

**Remark 3.2.1.** *every injective and  $e$ -injective semimodules are  $i$ -injective.*

**Lemma 3.2.8.** [3]

(1) *Let  $N$  be a left  $S$ -semimodule. Every retract of a left  $N$ - $e$ -injective  $S$ -semimodule is  $N$ - $e$ -injective.*

(2) *A retract of an  $e$ -injective  $S$ -semimodule is  $e$ -injective.*

*Proof.* need to prove the first part only:

Let  $B$  be an  $N - e - injective$  left  $S$ -semimodule and  $A$  a retract of  $B$  along with  $S$ -homomorphisms  $\lambda : A \longrightarrow B$  and  $\gamma : B \longrightarrow A$  such that  $\gamma \circ \lambda = id_A$ . Let  $\phi : Z \rightarrow N$  be a *normal  $S$ -monomorphism* and  $\beta : Z \rightarrow A$  be an  $S$ -homomorphism.

$$\begin{array}{ccccc}
0 & \longrightarrow & Z & \xrightarrow{\phi} & N \\
& & \downarrow \beta & & \swarrow \text{---} \eta^* \\
& & A & & \\
& & \downarrow \lambda & & \\
& & B & & 
\end{array}$$

Since  $B$  is  $N$ - $e$ -injective, there is an  $S$ -Homomorphism map  $\eta^* : N \rightarrow B$  such that  $\eta^* \circ \phi = \lambda \circ \beta$ . Consider  $\eta = \gamma \circ \eta^* : N \rightarrow A$  such that  $\eta \circ \phi = (\gamma \circ \eta^*) \circ \phi = \gamma \circ (\eta^* \circ \phi) = \gamma \circ (\lambda \circ \beta) = (\gamma \circ \lambda) \circ \beta = id_A \circ \beta = \beta$ . Suppose  $\eta' : N \rightarrow A$  is an  $S$ -homomorphism such that  $\eta' \circ \phi = \beta$ . Note that  $\lambda \circ \eta' \circ \phi = \lambda \circ \beta$ . Since  $B$  is  $N$ - $e$ -injective, there exist  $S$ -homomorphisms  $\eta_1^*, \eta_2^* : N \rightarrow B$  such that  $\eta_1^* \circ \phi = 0 = \eta_2^* \circ \phi$  and  $\eta^* + \eta_1^* = \lambda \circ \eta' + \eta_2^*$ .

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z & \xrightarrow{f} & N & \overset{\exists \eta_1^*}{\dashrightarrow} & B \\
& & \downarrow \beta & & \swarrow \eta & \overset{\exists \eta_2^*}{\dashrightarrow} & \\
& & A & & \swarrow \eta' & & \\
& & \downarrow \lambda & & \swarrow \text{---} \eta^* & & \\
& & B & & & & 
\end{array}$$

Consider  $\eta_1 = \gamma \circ \eta_1^*$  and  $\eta_2 = \gamma \circ \eta_2^*$ . Then we have for  $i = 1, 2$ ,  $\eta_i \circ \phi = \gamma \circ \eta_i^* \circ \phi = \gamma \circ 0 = 0$ . Moreover, we have  $\eta + \eta_1 = \gamma \circ \lambda \circ \eta' + \gamma \circ \eta_1^* = \gamma \circ (\lambda \circ \eta' + \eta_1^*) = \gamma \circ (\lambda \circ \eta' + \eta_2^*) = \gamma \circ \lambda \circ \eta' + \gamma \circ \eta_2^* = \eta' + \eta_2$ .

■

**Proposition 3.2.9.** [3] *Let  $N$  be a left  $S$ -semimodule and the set  $\{A_\lambda\}_{\lambda \in \Lambda}$  is a family of left  $S$ -semimodules, then  $\prod_{\lambda \in \Lambda}$  is  $N$ - $e$ -injective if and only if  $A_\lambda$  is  $N$ - $e$ -injective for any  $\lambda \in \Lambda$ .*

*Proof.* Let  $A = \prod_{\lambda \in \Lambda}$  and, for any  $\lambda \in \Lambda$ , let  $i_\lambda : A_\lambda \rightarrow A$  and  $\pi_\lambda : A \rightarrow A_\lambda$  be the canonical  $S$ -homomorphisms.

( $\implies$ ) For any  $\lambda \in \Lambda$ , we have  $\pi_\lambda \circ i_\lambda = id_{A_\lambda}$ , i.e.  $A_\lambda$  is a retract of  $A$ , so  $A_i$  is  $N$ - $e$ -injective (By lemma 3.2.8( .

( $\impliedby$ ) Suppose that  $A_\lambda$  is  $N$ - $e$ -injective for any  $\lambda \in \Lambda$ . Let  $\phi : Z \rightarrow N$  be a normal monomorphism and  $\beta : Z \rightarrow A$  an  $S$ -homomorphism.

$$\begin{array}{ccccc}
 0 & \longrightarrow & Z & \xrightarrow{\phi} & N \\
 & & \downarrow \beta & & \swarrow \text{---} \eta_\lambda^* \\
 & & A & & \\
 & & \downarrow \pi_\lambda & & \\
 & & A_\lambda & & 
 \end{array}$$

Since  $A_\lambda$  is  $N$ - $e$ -injective for any  $\lambda \in \Lambda$ , there is an  $S$ -homomorphism  $\eta_\lambda^* : N \rightarrow A_\lambda$  such that  $\eta_\lambda^* \circ \phi = \pi_\lambda \circ \beta$ . By the *Universal Property of Direct Products*, there exists an  $S$ -homomorphism

$$\eta : N \longrightarrow A, \quad n \mapsto \prod_{\lambda \in \Lambda} (i_\lambda \circ \eta_\lambda^*)(n).$$

Note that for any  $z \in Z$ , we have that

$$(\eta \circ \phi)(z) = \prod_{\lambda \in \Lambda} (i_\lambda \circ \eta_\lambda^*)(\phi(z)) = \prod_{\lambda \in \Lambda} (i_\lambda \circ \pi_\lambda)(\beta(z)) = \beta(z).$$

Assume that there exists an  $S$ -homomorphism  $\eta' : N \rightarrow A$  such that  $\eta' \circ \phi = \beta$ . It follows  $\pi_\lambda \circ \eta' \circ \phi = \pi_\lambda \circ \beta$  for any  $\lambda \in \Lambda$ . Since  $A_\lambda$  is

$N$ - $e$ -injective, then there exist  $S$ -homomorphisms  $\eta_{1_\lambda}^*, \eta_{2_\lambda}^* : N \rightarrow A$  such that  $\eta_{1_\lambda}^* \circ f = 0 = \eta_{2_\lambda}^* \circ \phi$  and  $\eta_\lambda^* + \eta_{1_\lambda}^* = \pi_\lambda \circ \eta' + \eta_{2_\lambda}^*$ .

$$\begin{array}{ccccc}
0 & \longrightarrow & Z & \xrightarrow{\phi} & N & \xrightarrow{\exists \eta_{1_\lambda}^*} & A_\lambda \\
& & \downarrow \beta & \swarrow \exists \eta & \downarrow & \swarrow \exists \eta_{2_\lambda}^* & \\
& & A & & & & \\
& & \downarrow \pi_\lambda & \swarrow \exists \eta' & \downarrow & \swarrow \exists \eta_\lambda^* & \\
& & A_\lambda & & & & 
\end{array}$$

For  $i = 1, 2$ , there exists by the *Universal Property of Direct Products* an  $S$ -homomorphism

$$\eta_i : N \longrightarrow A, \quad n \mapsto \prod_{\lambda \in \Lambda} (i_\lambda \circ \eta_{i_\lambda}^*)(n).$$

For  $i = 1, 2$  and any  $z \in Z$  we have that  $(\eta_i \circ \phi)(z) = \prod_{\lambda \in \Lambda} (i_\lambda \circ \eta_{i_\lambda}^*)(\phi(z)) = \prod_{\lambda \in \Lambda} i_\lambda(0) = 0$ . Moreover, we have for any  $n \in N$  :

$$\begin{aligned}
(\eta + \eta_1)(n) &= \prod_{\lambda \in \Lambda} (i_\lambda \circ \pi_\lambda \circ \eta)(n) + \prod_{\lambda \in \Lambda} (i_\lambda \circ \eta_{1_\lambda}^*)(n) \\
&= \prod_{\lambda \in \Lambda} (i_\lambda \circ (\pi_\lambda \circ \eta + \eta_{1_\lambda}^*))(n) \\
&= \prod_{\lambda \in \Lambda} (i_\lambda \circ (\eta_\lambda^* + \eta_{1_\lambda}^*))(n) \\
&= \prod_{\lambda \in \Lambda} (i_\lambda \circ (\pi_\lambda \circ \eta' + \eta_{2_\lambda}^*))(n) \\
&= \prod_{\lambda \in \Lambda} (\eta' + i_\lambda \circ \eta_{2_\lambda}^*)(n) \\
&= (\eta' + \eta_2)(n).
\end{aligned}$$

■

# Chapter 4

## Future work

We will be working on making a characterization of *weekly projective semimodule*, it will take into consideration all the aspects presented in *the weekly projective module*. which is defined as the following:

- Recall that *An epimorphism  $p : P \longrightarrow M$  is called a **projective cover** of  $M$  if and only if  $P$  is projective and  $p$  is a small epimorphism ( $\ker(p)$  is small in  $P$ ). We denote projective cover of  $M$  by  $P(M)$ .*
- Recall that *A module  $M$  is called **weakly projective** if and only if has a projective cover  $\alpha : P(M) \longrightarrow M$  and every map from  $P(M)$  into a finitely generated (free) module can be factored through  $M$  via an epimorphism (not nonsecretly equal  $\alpha$ )*

Therefore, the proposed definition of weekly projective semimodule will be mainly dependent on the definition of the *projective cover* of *semimodule* (Def.3.1.5). And it will be as follows:

*An  $S$ -semimodule  $M$  is **weakly projective semimodule** if and only if has a projective cover  $\alpha : P(M) \longrightarrow M$  and every map from  $P(M)$  into a finitely generated semimodule can be factored through  $M$  via an*

*surjective  $S$ -homomorphism (not nonsecretly equal  $\alpha$ ).*

The expected difference between the two definitions (*weakly projective module and the proposed definition of weakly projective semimodule*) will be in the nature of definitions of modules and semimodules.

After we proposed our definition, we will make sure that this it is not another definition of *semimodule*, and it will be by using a counter example to prove that every *projective semimodule is weakly projective semimodule* but the converse is not.

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